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# A Multi-Criteria Group Decision-Making Approach for Robot Selection Using Interval-Valued Intuitionistic Fuzzy Information And Aczel-Alsina Bonferroni Means

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**ABSTRACT**

The process of identifying the most appropriate robot for a particular industrial task has grown challenging and more difficult in the fast-paced environment. It is merely driven by the complex evolution and continuous integration of modern characteristics and advanced features by various suppliers. Industrial robots are now widely available in the marketplace, each possessing a distinctive collection of skills, attributes, and requirements. However, the selection of optimal robots is heavily influenced by factors such as the manufacturing environment, product design, production system, and overall cost considerations. These factors directly impact the decision-making process. The ultimate goal for the decision-maker is to pinpoint and choose the most suitable robot, capable of delivering the desired output while minimizing costs and catering to the specific requirements of the industry. So, to consider this, in this paper, the hybrid structure of the Aczel–Alsina (AA) and Bonferroni mean (BM) operators for the interval-valued intuitionistic fuzzy (IVIF) environment has been proposed, which can show the interrelationship between multiple criteria and assist experts in the decision-making (DM) process. Moreover, the algorithm and methodology for the multi-criteria group decision-making (MCGDM) problem have been defined, which are further utilized by solving a real-world problem to demonstrate the effectiveness and validity of the proposed method. At last, the comparative analysis between prior and proposed studies has been presented, followed by the conclusion of the results.

## 1. Introduction

In 2012, ISO-Standard 8373 defined a robot as a programmable, actuated device that is capable of moving independently, in two or more directions within its surroundings to execute predefined activities [1]. So, an industrial robot is considered a programmable manipulating device that acts as

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a bridge between the automated machine designed for carrying out specific tasks and laborious tasks which are capable of performing a wide range of job functions [2] while the other features of industrial robots contribute to its overall utility in which it included its DM ability, its ability to interact with other machines, and its potential to react different sensory information. All these features contribute to it being a crucial instrument across a wide spectrum of industrial tasks such as handling material, assembling components, completing finishing work, machine loading, applying spray paint, and conducting welding. When selecting an industrial robot for a specific task, certain key factors must be carefully considered in which it includes control resolution, the accuracy of the robot, repeatability, load-carrying capacity, degrees of freedom, interfacing of human-machine capabilities, programming flexibility, maximum tip speed, memory capacity, and the quality of service provided by the supplier. All these features are classified into cost type and benefit type attributes which means that all quantifiable features such as cost, and load-carrying capacity are included in cost type while others such as memory capacity, programming flexibility, etc. belong to benefit type attributes. The selection of an industrial robot for a specific task requires a comprehensive evaluation of these attributes as it involves striking a balance between these attributes and the desired performance of the robot. Numerous methodologies for robot selection have been proposed by expertise in the past in which it includes the application of multi-criteria decision-making (MCDM) techniques [3], optimization models for production system performance [4], computer-assisted models [5], and statistical models [6] etc. Many experts such as Magrini *et al.,* [7] proposed the coexistence of human robots and their interaction, and Javaid *et al.,* [8] defined the substantial capabilities of robots.

In real-life scenarios, we often encounter situations marked by vagueness, imprecision, and ambiguity in which the effective DM demands a systematic approach, involving collection of data, risk assessment, consultation with experts, and the ability to adapt as circumstances unfold. To deal with such kind of vagueness and uncertainties, Zadeh [9] introduced the fuzzy set (FS) in 1965 which can deal with the membership value (MV) of the objects and assists the expertise in the DM problem by reducing the vagueness of objects but in everyday life, we encounter both positive and negative aspects of the various object which make this theory unreliable. So, by considering this, Atanassov [10] introduced the concept of intuitionistic fuzzy set (IFS) which can deal with the MV and non-membership value (NMV) of the object and provides the expertise and more reliable theory that helps them in DM problem in a more precise manner but also has some limitations i.e. sum of its MV and NMV is lying in between [0, 1]. As time goes on, the expertise faces challenges to deal with the problems that are defined by the MV and NMV as sometimes, it cannot characterized by quasi-exact values [11]. So this theory was extended into the interval-valued intuitionistic fuzzy set (IVIFS) by Atanassov [12] which makes it more enhanced as it deals with the duplet over the interval and the sum of its upper bounds in [0,1].

In the context of MCGDM problems, there are generally two principal methodologies involved. The first approach involves the use of conventional techniques such as TOPSIS, VIKOR, ELECTRE, etc. These conventional methods have been established over time and are commonly used for DM where multiple criteria need to be considered. The other one, an alternative, and more efficient approach involves the use of Aggregation Operators (AOs) for handling the DM challenges. AOs have demonstrated superiority over traditional methods for several reasons. Unlike traditional techniques that primarily yield ranking results for the alternatives, AOs have the distinct advantage of providing complete values for each of the alternatives in addition to the ranking outcomes. This capability of offering both ranking and precise quantitative values for alternatives is a valuable feature, as this comprehensive information enables the expertise to make decisions more precisely in complex decision scenarios. Moreover, recent research and academic endeavors have led to a heightened focus on AOs within the MCGDM domain. Researchers and scholars have recognized the potential of

AOs to contribute significantly to the understanding and resolution of complex decision problems involving multiple criteria. Detyniecki [13] introduced the fundamentals of AOs, Yager [14] introduced the concept of prioritized AOs, Akram *et al.,* [15] proposed the prioritized weighted AO, Senapati *et al.,* [16] proposed the AA AOs, Zhang *et al.,* [17] introduced the Frank power (FP) AOs, Seikh *et al.,* [18] defined the Dombi AO, Liu [19] proposed the Hamacher AO, Irvanizam *et al.,* [20] proposed the harmonic and arithmetic mean operator, Rahim *et al.,* [21] proposed quasring AO, Dong *et al.,* [22] suggested the family of AO. Further, Yao and Guo defined [23] AO for decision-making analysis, Hezam *et al.*, [24] defined the geometric AO, and Attaullah *et al.*, [25] proposed the Fermantean AO for hesitant rough values. However, all these operators are unable to show the relation between the multiple criteria in the DM problems. So, for this, Yager [26] introduced the AO which is named as Bonferroni mean operator (BM) can show the relationship between the multiple criteria. Further BM is extended into power Bonferroni mean AO [27], Bonferroni prioritized AO [28], and geometric Bonferroni mean operator (GBM) [29], etc.

MCGDM is indeed a widely applicable problem-solving framework encountered in various disciplines including business and management, medicines and healthcare, engineering, social sciences, information technologies, political sciences, etc. It arises in situations where decision-makers need to consider multiple conflicting criteria to make precise and balanced decisions. So, to consider it, multiple researchers utilize different t-norms with the BM AO to resolve the multiple conflicting criteria including [30-34] but all these approaches do not define the information within the interval. So, by considering this, in this paper, we will introduce the concept of AABM AO which is the hybrid structure and form by combining the AA and BM AOs for the IVIF set and gives the more efficient and precise results. Briefly, the purpose of this paper appears as follows:

i. To introduce a novel approach to MCGDM by using the AABMAO for the IVIF set.
ii. To explore fundamental characteristics and attributes of the suggested theory and determine the suitable approach for managing DM algorithms.
iii. To illustrate that this theory enhances adaptability and efficiency as it is a hybrid structure formed by combining the AA and BM theory and shows the most desirable outcomes.
iv. To demonstrate its superiority as the most versatile approach through a comparative analysis with existing information and discuss a thorough examination of the proposed work's validity.
v. To provide a comprehensive analysis of the proposed approach's properties, efficiency, and flexibility compared to existing methods.

The paper outlines its structure, indicating that some basic definitions and fundamental operators are reviewed in Section 2. In section 3, we introduced AABM AO for the IVIF environment, along with theorems and properties, section 4 consists of an MCGDM algorithm and methodologies defined for solving the DM problem, and the illustration of a real-world application to validate the proposed work, followed by a comparative analysis with prior theories is discussed in Section 5 and 6 respectively.

## 2. Preliminaries

In this section, we will discuss some fundamental concepts of the IVIF set and discuss some of its properties. For a better understanding of this paper, we will also discuss the t-norm, t-conorm, AA, and BM operators.

*Definition 1* [35]: Let $\mathbb{U}$ be the universe of discourse and $\alpha \in \mathbb{U}$. Then, an IVIFS $\Upsilon$ on $\mathbb{U}$ is defined as:

$$\Upsilon: \{\langle \alpha, (\mathbb{t}_{\Upsilon}(\alpha), \mathbb{f}_{\Upsilon}(\alpha)) | \alpha \in \mathbb{U} \rangle\} \tag{1}$$

Where $\mathbb{t}_{\Upsilon}(\alpha)$ and $\mathbb{f}_{\Upsilon}(\alpha)$ are in the form of closed interval whose upper and lower bound is denoted by $\mathbb{t}_{\Upsilon}(\alpha)^{+}$, $\mathbb{t}_{\Upsilon}(\alpha)^{-}$, $\mathbb{f}_{\Upsilon}(\alpha)^{+}$ *and* $\mathbb{f}_{\Upsilon}(\alpha)^{-}$ and defined as:

$$\Upsilon = \{\alpha, ([\mathbb{t}_{\Upsilon}(\alpha)^{-}, \mathbb{t}_{\Upsilon}(\alpha)^{+}], [\mathbb{f}_{\Upsilon}(\alpha)^{+}, \mathbb{f}_{\Upsilon}(\alpha)^{-}]) | \alpha \in \mathbb{U}\} \quad (2)$$

Such that $0 < \mathbb{t}_{\Upsilon}(\alpha)^{+} + \mathbb{f}_{\Upsilon}(\alpha)^{+} \leq 1$ and $\mathbb{t}_{\Upsilon}(\alpha)^{-}, \mathbb{f}_{\Upsilon}(\alpha)^{-} \geq 0$.

For $\alpha \in \mathbb{U}$, we can also compute the hesitancy degree $\hbar_{\Upsilon}(\alpha)$ of an IVIFS, defined as:

$$\hbar_{\Upsilon}(\alpha) = 1 - \mathbb{t}_{\Upsilon}(\alpha) - \mathbb{f}_{\Upsilon}(\alpha) = [1 - \mathbb{t}_{\Upsilon}(\alpha)^{+} - \mathbb{f}_{\Upsilon}(\alpha)^{+}, 1 - \mathbb{t}_{\Upsilon}(\alpha)^{-} - \mathbb{f}_{\Upsilon}(\alpha)^{-}] \quad (3)$$

*Definition 2* [35]: Consider $\Upsilon_{j} = \left(\mathbb{t}_{\Upsilon j}(\alpha), \mathbb{f}_{\Upsilon j}(\alpha)\right); (j = 1,2, \dots, N)$ be a two IVIFS s.t $\mathbb{t}_{\Upsilon j}(\alpha) = \left[\mathbb{t}_{\Upsilon j}(\alpha)^{-}, \mathbb{t}_{\Upsilon j}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon j}(\alpha) = \left[\mathbb{f}_{\Upsilon j}(\alpha)^{-}, \mathbb{f}_{\Upsilon j}(\alpha)^{+}\right]$, then

1) $$\Upsilon_1 \oplus \Upsilon_2 = \left(\begin{matrix}\begin{bmatrix}\mathbb{t}_{\Upsilon 1}(\alpha)^{-} + \mathbb{t}_{\Upsilon 2}(\alpha)^{-} - \mathbb{t}_{\Upsilon 1}(\alpha)^{-}.\mathbb{t}_{\Upsilon 2}(\alpha)^{-}, \\ \mathbb{t}_{\Upsilon 1}(\alpha)^{+} + \mathbb{t}_{\Upsilon 2}(\alpha)^{+} - \mathbb{t}_{\Upsilon 1}(\alpha)^{+}.\mathbb{t}_{\Upsilon 2}(\alpha)^{+}\end{bmatrix}, \\ [\mathbb{f}_{\Upsilon 1}(\alpha)^{-}.\mathbb{f}_{\Upsilon 2}(\alpha)^{-}, \mathbb{f}_{\Upsilon 1}(\alpha)^{+}.\mathbb{f}_{\Upsilon 2}(\alpha)^{+}]\end{matrix}\right) \quad (1)$$
2) $$\Upsilon_1 \otimes \Upsilon_2 = \left(\begin{matrix}[\mathbb{t}_{\Upsilon 1}(\alpha)^{-}.\mathbb{t}_{\Upsilon 2}(\alpha)^{-}, \mathbb{t}_{\Upsilon 1}(\alpha)^{+}.\mathbb{t}_{\Upsilon 2}(\alpha)^{+}], \\ \begin{bmatrix}\mathbb{f}_{\Upsilon 1}(\alpha)^{-} + \mathbb{f}_{\Upsilon 2}(\alpha)^{-} - \mathbb{f}_{\Upsilon 1}(\alpha)^{-}.\mathbb{f}_{\Upsilon 2}(\alpha)^{-}, \\ \mathbb{f}_{\Upsilon 1}(\alpha)^{+} + \mathbb{f}_{\Upsilon 2}(\alpha)^{+} - \mathbb{f}_{\Upsilon 1}(\alpha)^{+}.\mathbb{f}_{\Upsilon 2}(\alpha)^{+}\end{bmatrix}\end{matrix}\right) \quad (2)$$
3) $$\kappa\Upsilon_1 = ([1 - (1 - \mathbb{t}_{\Upsilon 1}(\alpha)^{-})^{\kappa}, 1 - (1 - \mathbb{t}_{\Upsilon 1}(\alpha)^{+})^{\kappa}], [(\mathbb{f}_{\Upsilon 1}(\alpha)^{-})^{\kappa}, (\mathbb{f}_{\Upsilon 1}(\alpha)^{+})^{\kappa}]) \quad (3)$$
4) $$\Upsilon_1{}^{\kappa} = ([(\mathbb{t}_{\Upsilon 1}(\alpha)^{-})^{\kappa}, (\mathbb{t}_{\Upsilon 1}(\alpha)^{+})^{\kappa}], [1 - (1 - \mathbb{f}_{\Upsilon 1}(\alpha)^{-})^{\kappa}, 1 - (1 - \mathbb{f}_{\Upsilon 1}(\alpha)^{+})^{\kappa}]) \quad (4)$$

*Definition 3* [36]: Consider $\Upsilon_1 = \left(\mathbb{t}_{\Upsilon 1}(\alpha), \mathbb{f}_{\Upsilon 1}(\alpha)\right)$ s.t $\mathbb{t}_{\Upsilon 1}(\alpha) = [\mathbb{t}_{\Upsilon 1}(\alpha)^{-}, \mathbb{t}_{\Upsilon 1}(\alpha)^{+}]$ and $\mathbb{f}_{\Upsilon 1}(\alpha) = [\mathbb{f}_{\Upsilon 1}(\alpha)^{-}, \mathbb{f}_{\Upsilon 1}(\alpha)^{+}]$ be an IVIFS, then the score function, accuracy function $\mathcal{A}_{\Upsilon}$ and hesitancy degree index $\mathcal{H}_{\Upsilon}$ is defined as:

$$\mathcal{S}_{\Upsilon} = \frac{1}{2}(\mathbb{t}_{\Upsilon 1}(\alpha)^{-} + \mathbb{t}_{\Upsilon 1}(\alpha)^{+} - \mathbb{f}_{\Upsilon 1}(\alpha)^{-} - \mathbb{f}_{\Upsilon 1}(\alpha)^{+}) \in [-1,1] \quad (5)$$

$$\mathcal{A}_{\Upsilon} = \frac{1}{2}(\mathbb{t}_{\Upsilon 1}(\alpha)^{-} + \mathbb{t}_{\Upsilon 1}(\alpha)^{+} - \mathbb{f}_{\Upsilon 1}(\alpha)^{-} - \mathbb{f}_{\Upsilon 1}(\alpha)^{+}) \in [-1,1] \quad (6)$$

$$\mathcal{H}_{\Upsilon} = (\mathbb{t}_{\Upsilon 1}(\alpha)^{+} + \mathbb{f}_{\Upsilon 1}(\alpha)^{+} - \mathbb{t}_{\Upsilon 1}(\alpha)^{-} - \mathbb{f}_{\Upsilon 1}(\alpha)^{-}) \in [-1,1] \quad (7)$$

C. Bonferroni was the first who define the BM operator which is used to indicate the interrelation between the different values.

*Definition 4* [37]*:* Consider $\Upsilon_{j} = \left(\mathbb{t}_{\Upsilon j}(\alpha), \mathbb{f}_{\Upsilon j}(\alpha)\right); (j = 1,2, \dots, N)$ consists of $\mathbb{R}^{+}$and $\ell, m > 0$. Then, the AO is defined as:

$$BM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \left(\frac{1}{N(N-1)} \oplus_{\substack{p,q=1 \\ p \neq q}}^{N} \Upsilon_p^{\ell} \Upsilon_q^{m}\right)^{\frac{1}{\ell + m}} \quad (8)$$

Which is called the BM operator.

*Definition 5* [38]*:* Consider $\Upsilon_{j} = \left(\mathbb{t}_{\Upsilon j}(\alpha), \mathbb{f}_{\Upsilon j}(\alpha)\right)$ s.t $\mathbb{t}_{\Upsilon j}(\alpha) = \left[\mathbb{t}_{\Upsilon j}(\alpha)^{-}, \mathbb{t}_{\Upsilon j}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon j}(\alpha) = \left[\mathbb{f}_{\Upsilon j}(\alpha)^{-}, \mathbb{f}_{\Upsilon j}(\alpha)^{+}\right]; (j = 1,2, \dots, N)$ consists of $\mathbb{R}^{+}$and $\ell, m > 0$. Then,

$$IVIFBM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \left(\frac{1}{N(N-1)}\left(\oplus_{\substack{p,q=1 \\ p \neq q}}^{N} \left(\Upsilon_p^{\ell} \otimes \Upsilon_q^{m}\right)\right)\right)^{\frac{1}{\ell + m}} \quad (9)$$

Which is called the interval-valued intuitionistic fuzzy Bonferroni mean (IVIFBM) operator.

### *2.1 Aczel-Alsina Operations of IVIFS*

Aczel and Alsina introduced the concept of AA t-norm $\left(\mathbb{T}_{\mathrm{p}}^{\mathfrak{z}}\right)_{\mathfrak{z} \in [0,\infty]}$ and t-conorm $\left(\mathbb{S}_{\mathrm{p}}^{\mathfrak{z}}\right)_{\mathfrak{z} \in [0,\infty]}$ which is defined as:

*Definition 6* [39]*:* AA t-norm $\left(\mathbb{T}_{\mathrm{p}}^{\mathfrak{z}}\right)_{\mathfrak{z} \in [0,\infty]}$ is determined by

$$\mathbb{T}_{\mathrm{p}}^{\mathfrak{z}}(\mathbb{x},\mathbb{y})=\begin{cases}\mathbb{T}_D(\mathbb{x},\mathbb{y}) & ;\ \mathfrak{z}=0\\ min(\mathbb{x},\mathbb{y}) & ;\mathfrak{z}=\infty\\ \mathbb{e}xp^{-((-\ln(\mathbb{x}))^{\mathfrak{z}}+(-\ln(\mathbb{y}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}} & ;\ otherwise\end{cases} \tag{10}$$

When $\mathfrak{z}=1$, it becomes an algebraic t-norm.
And
AA t-conorm $\left(\mathbb{S}_{\mathrm{p}}^{\mathfrak{z}}\right)_{\mathfrak{z}\in[0,\infty]}$ is determined by

$$\mathbb{S}_{\mathrm{p}}^{\mathfrak{z}}(\mathbb{x},\mathbb{y})=\begin{cases}\mathbb{S}_D(\mathbb{x},\mathbb{y}) & ;\ \mathfrak{z}=0\\ \vee(\mathbb{x},\mathbb{y}) & ;\mathfrak{z}=\infty\\ 1-\mathbb{e}xp^{-((-\ln(1-\mathbb{x}))^{\mathfrak{z}}+(-\ln(1-\mathbb{y}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}} & ;\ otherwise\end{cases} \tag{11}$$

When $\mathfrak{z}=1$, it becomes algebraic t-conorm and $\forall\,\mathfrak{z}\in[0,\infty]$, $\mathbb{T}_{\mathrm{p}}^{\mathfrak{z}}$ and $\mathbb{S}_{\mathrm{p}}^{\mathfrak{z}}$are the dual of each other.

*Definition 7* [16]*:* Consider $\Upsilon_1=\left(\mathbb{t}_{\Upsilon 1}(\alpha),\mathbb{f}_{\Upsilon 1}(\alpha)\right)$ and $\Upsilon_2=\left(\mathbb{t}_{\Upsilon 2}(\alpha),\mathbb{f}_{\Upsilon 2}(\alpha)\right)$ be a two IVIFS, s.t $\mathfrak{z}\geq 1$; $\mu>0$ and $\mathbb{t}_{\Upsilon_{j}}(\alpha)=\left[\mathbb{t}_{\Upsilon_{j}}(\alpha)^{-},\mathbb{t}_{\Upsilon_{j}}(\alpha)^{+}\right]$; $\mathbb{f}_{\Upsilon_{j}}(\alpha)=\left[\mathbb{f}_{\Upsilon_{j}}(\alpha)^{-},\mathbb{f}_{\Upsilon_{j}}(\alpha)^{+}\right]$. Then, the operations of $\mathbb{T}_{\mathrm{p}}^{\mathfrak{z}}$ and $\mathbb{S}_{\mathrm{p}}^{\mathfrak{z}}$ is defined as:

i. $$\Upsilon_1\oplus\Upsilon_2=\begin{pmatrix}\begin{bmatrix}1-\mathbb{e}xp^{-((-\ln(1-\mathbb{t}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}}+(-\ln(1-\mathbb{t}_{\Upsilon 2}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\\ 1-\mathbb{e}xp^{-((-\ln(1-\mathbb{t}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}}+(-\ln(1-\mathbb{t}_{\Upsilon 2}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\end{bmatrix}\\ \begin{bmatrix}\mathbb{e}xp^{-((-\ln(\mathbb{f}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}}+(-\ln(\mathbb{f}_{\Upsilon 2}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\\ \mathbb{e}xp^{-((-\ln(\mathbb{f}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}}+(-\ln(\mathbb{f}_{\Upsilon 2}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\end{bmatrix}\end{pmatrix} \tag{12}$$

Where $\mathbb{e}xp$ represents the exponential function.

ii. $$\Upsilon\otimes\Upsilon_1=\begin{pmatrix}\begin{bmatrix}\mathbb{e}xp^{-((-\ln(\mathbb{t}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}}+(-\ln(\mathbb{t}_{\Upsilon 2}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\mathbb{e}xp^{-((-\ln(\mathbb{t}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}}+(-\ln(\mathbb{t}_{\Upsilon 2}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\end{bmatrix}\\ \begin{bmatrix}1-\mathbb{e}xp^{-((-\ln(1-\mathbb{f}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}}+(-\ln(1-\mathbb{f}_{\Upsilon 2}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\\ 1-\mathbb{e}xp^{-((-\ln(1-\mathbb{f}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}}+(-\ln(1-\mathbb{f}_{\Upsilon 2}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\end{bmatrix}\end{pmatrix} \tag{13}$$

iii. $$\mu\Upsilon_1=\begin{pmatrix}\left[1-\mathbb{e}xp^{-(\mu(-\ln(1-\mathbb{t}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},1-\mathbb{e}xp^{-(\mu(-\ln(1-\mathbb{t}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right],\\ \left[\mathbb{e}xp^{-(\mu(-\ln(\mathbb{f}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\mathbb{e}xp^{-(\mu(-\ln(\mathbb{f}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right]\end{pmatrix} \tag{14}$$

iv. $$\Upsilon_1{}^{\mu}=\begin{pmatrix}\left[\mathbb{e}xp^{-(\mu(-\ln(\mathbb{t}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\mathbb{e}xp^{-(\mu(-\ln(\mathbb{t}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right],\\ \left[1-\mathbb{e}xp^{-(\mu(-\ln(1-\mathbb{f}_{\Upsilon 1}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},1-\mathbb{e}xp^{-(\mu(-\ln(1-\mathbb{f}_{\Upsilon 1}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right]\end{pmatrix} \tag{15}$$

*Theorem 1:* Consider $\Upsilon_{j}=\left(\mathbb{t}_{\Upsilon_{j}}(\alpha),\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)$; $(j=1,2,\ldots,N)$ be a collection of IVIFSs, s.t $\mathbb{t}_{\Upsilon_{j}}(\alpha)=\left[\mathbb{t}_{\Upsilon_{j}}(\alpha)^{-},\mathbb{t}_{\Upsilon_{j}}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon_{j}}(\alpha)=\left[\mathbb{f}_{\Upsilon_{j}}(\alpha)^{-},\mathbb{f}_{\Upsilon_{j}}(\alpha)^{+}\right]$. Then,

i. $\Upsilon \oplus \Upsilon_1 = \Upsilon_1 \oplus \Upsilon$ (16)
ii. $\Upsilon \otimes \Upsilon_1 = \Upsilon_1 \otimes \Upsilon$ (17)
iii. $\mu(\Upsilon \oplus \Upsilon_1) = \mu\Upsilon_1 \oplus \varphi\Upsilon , \mu > 0$ (18)
iv. $(\mu_1 + \mu_2)\Upsilon = \mu_1\Upsilon + \mu_2\Upsilon, \mu_1, \mu_2 > 0$ (19)
v. $(\Upsilon \otimes \Upsilon_1)^{\mu} = (\Upsilon)^{\mu} \otimes (\Upsilon_1)^{\mu}, \mu > 0$ (20)
vi. $\Upsilon^{\mu 1} \otimes \Upsilon^{\mu 2} = \Upsilon^{(\mu 1+\mu 2)}, \mu_1, \mu_2 > 0$ (21)

## 3. Aczel-Alsina Bonferroni Mean Operator based on IVIFS

In this section, we present Interval-valued Intuitionistic fuzzy Aczel-Alsina Bonferroni mean (IVIFAABM) averaging AOs by using AA operations.

*Theorem 2:* (*Idempotency*) Consider $\Upsilon_{j} = \left(\mathbb{t}_{\Upsilon j}(\alpha), \mathbb{f}_{\Upsilon j}(\alpha)\right); (j = 1,2, \dots, N)$ be a two IVIFS s.t $\mathbb{t}_{\Upsilon j}(\alpha) = \left[\mathbb{t}_{\Upsilon j}(\alpha)^{-}, \mathbb{t}_{\Upsilon j}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon j}(\alpha) = \left[\mathbb{f}_{\Upsilon j}(\alpha)^{-}, \mathbb{f}_{\Upsilon j}(\alpha)^{+}\right]$, $\mathfrak{z} \geq 1$ and $\mu > 0$. Then, the accumulated value by using IVIFAABM is also an IVIFS.

$$IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \left(\frac{1}{N\,(N-1)}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N}\left(\Upsilon_p^{\ell} \otimes \Upsilon_q^{m}\right)\right)\right)^{\frac{1}{\ell+m}} \tag{22}$$

*Proof*: Let $\Upsilon_p^{\ell}$ and $\Upsilon_q^{m}$ be the two IVIFS. Then, by using definition 7;

$$\Upsilon_p^{\ell} = \left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(\ell(-\ln(1-\mathbb{f}_{\Upsilon p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)$$

$$= \left(\begin{array}{c}\left[\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon p}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, \mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon p}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right], \\ \left[1 - \mathbb{e}xp^{-(\ell(-\ln(1-\mathbb{f}_{\Upsilon p}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(\ell(-\ln(1-\mathbb{f}_{\Upsilon p}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right]\end{array}\right)$$

$$\Upsilon_q^{m} = \left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(m(-\ln(1-\mathbb{f}_{\Upsilon q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)$$

$$= \left(\begin{array}{c}\left[\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon q}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, \mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon q}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right], \\ \left[1 - \mathbb{e}xp^{-(m(-\ln(1-\mathbb{f}_{\Upsilon q}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(m(-\ln(1-\mathbb{f}_{\Upsilon q}(\alpha)^{+}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right]\end{array}\right)$$

$$\Upsilon_p^{\ell} \otimes \Upsilon_q^{m} = \left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(\ell(-\ln(1-\mathbb{f}_{\Upsilon p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)$$
$$\otimes \left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}, 1 - \mathbb{e}xp^{-(m(-\ln(1-\mathbb{f}_{\Upsilon q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)$$

$$=\left(\begin{array}{c}\left[\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\wp}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}},\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\wp}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right],\\ \left[1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{q}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}},1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{q}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right]\end{array}\right)$$

$$\oplus_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(\Upsilon_{\wp}^{\ell}\otimes\Upsilon_{q}^{m}\right)=\left(\begin{array}{c}\left[\begin{array}{c}1-\left(\mathbb{e}xp^{-\left(\sum_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\wp}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}}\right)^{\frac{1}{\mathfrak{z}}},\\ 1-\left(\mathbb{e}xp^{-\left(\sum_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\wp}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}}\right)^{\frac{1}{\mathfrak{z}}}\end{array}\right],\\ \left[\begin{array}{c}\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{q}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}},\\ \mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{q}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\end{array}\right]\end{array}\right)$$

$$\frac{1}{N\,(N\;-1)}\left(\oplus_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(\Upsilon_{\wp}^{\ell}\otimes\Upsilon_{q}^{m}\right)\right)=\left(S_{1},T_{1}\right)$$

Where

$$S_1 = \Bigg(\Bigg[1 - \mathbb{e}xp\left(-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\left(\mathbb{e}xp\left(-\left(\sum_{\substack{p,q=1\\ p\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{p}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

$$1 - \mathbb{e}xp\left(-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\left(\mathbb{e}xp\left(-\left(\sum_{\substack{p,q=1\\ p\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{p}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\Bigg],$$

$$T_1=\left[\mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^-\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{q}}(\alpha)^-\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}},\ \mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\wp}}(\alpha)^+\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{q}}(\alpha)^+\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right]\Bigg)$$

$$\left(\frac{1}{N\,(N\,-1)}\left(\bigoplus_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(\Upsilon_{\wp}^{\ell}\otimes\ \Upsilon_{q}^{m}\right)\right)\right)^{\frac{1}{\ell+m}}=([S_2^-,S_2^+\ ],[T_2^-,T_2^+])$$

Where

$$S_2^-=\mathbb{e}xp^{-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(1-1-\left(\mathbb{e}xp^{-\left(\sum_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\wp}}(\alpha)^-\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{q}}(\alpha)^-\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}$$

$$S_2^{+} = \mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(1-1-\left(\mathbb{e}xp^{-\left(\sum_{\substack{\mathscr{p},\mathscr{q}=1\\ \mathscr{p}\neq\mathscr{q}}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{\mathscr{p}}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_{\mathscr{q}}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

$$T_2^{-} = 1-\mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\mathscr{p}}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{\mathscr{q}}}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

$$T_2^{+} = 1-\mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp^{-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{\mathscr{p}}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_{\mathscr{q}}}(\alpha)^{+}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)$$

*Remark*: Idempotency property does not hold for *Theorem 1*.

Theorem 3: (Monotonicity) Consider $\Upsilon_{j} = \left(\mathbb{t}_{\Upsilon_{j}}(\alpha), \mathbb{f}_{\Upsilon_{j}}(\alpha)\right)$ and $\Upsilon_{j}{}' = \left(\mathbb{t}_{\Upsilon_{j}}(\alpha)', \mathbb{f}_{\Upsilon_{j}}(\alpha)'\right)$; $(j = 1,2,\ldots,N)$ be a two IVIF sets s.t $\mathbb{t}_{\Upsilon_{j}}(\alpha) = \left[\mathbb{t}_{\Upsilon_{j}}(\alpha)^{-}, \mathbb{t}_{\Upsilon_{j}}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon_{j}}(\alpha) = \left[\mathbb{f}_{\Upsilon_{j}}(\alpha)^{-}, \mathbb{f}_{\Upsilon_{j}}(\alpha)^{+}\right]$ where $\mathbb{t}_{\Upsilon_{j}}(\alpha) \leq \mathbb{t}_{\Upsilon_{j}}(\alpha)'$ ; $\mathbb{f}_{\Upsilon_{j}}(\alpha) \geq \mathbb{f}_{\Upsilon_{j}}(\alpha)'$; $\forall\, j$, then

$$IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N) \leq IVIFAABM^{\ell,m}(\Upsilon_1{}', \Upsilon_2{}', \ldots, \Upsilon_N{}') \tag{23}$$

*Proof*: As $\mathbb{t}_{\Upsilon_{j}}(\alpha) \leq \mathbb{t}_{\Upsilon_{j}}(\alpha)'$ and $\mathbb{f}_{\Upsilon_{j}}(\alpha) \geq \mathbb{f}_{\Upsilon_{j}}(\alpha)'$ $\forall\, j$, then

$$\otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}$$

$$\geq \otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}$$

$$1-\left(\otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)$$

$$\leq 1-\left(\otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\begin{pmatrix}1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right) \\ \left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\end{pmatrix}^{\frac{1}{N(N-1)}}\right)$$

$$\left(1-\left(\otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}} \leq$$

$$\left(1-\left(\otimes_{\substack{p,q=1 \\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_p}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{j_q}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}$$

$$\left(1-1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{j_p}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{j_q}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m} \leq$$

$$\left(1-1-\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{j_p}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{j_q}}{}'(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}, \forall\, j$$

$$\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}} \geq$$

$$\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}$$

$$\left(1-\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}} \leq$$

$$\left(1-\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$1-\left(1-\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}} \geq$$

$$1-\left(1-\bigotimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_{p}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_{q}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\geq$$

$$1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_p}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon j_q}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon j_p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon j_q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}-$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_p}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_q}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)\leq$$

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon j_p}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon j_q}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}-$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon j_p}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon j_q}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)$$

Now, consider $\Upsilon_j = IVIFBM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N)$ and $\Upsilon_j{}' = IVIFBM^{\ell,m}(\Upsilon_1{}', \Upsilon_2{}', \dots, \Upsilon_N{}')$ be a IVIFS and the score value of $\Upsilon_j$ is $\mathcal{S}_{\Upsilon_j}$ and $\Upsilon_j{}'$ is $\mathcal{S}_{\Upsilon_j}{}'$ s.t $\mathcal{S}_{\Upsilon_j} \leq \mathcal{S}_{\Upsilon_j}{}'$. Then consider the following cases:

1) If $\mathcal{S}_{\Upsilon_j} \leq \mathcal{S}_{\Upsilon_j}{}'$, then

$$IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N) < IVIFAABM^{\ell,m}(\Upsilon_1{}', \Upsilon_2{}', \ldots, \Upsilon_N{}') \tag{24}$$

2) If $\mathcal{S}_{\Upsilon_j} = \mathcal{S}_{\Upsilon_j}{}'$, then

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{jp}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{jq}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}} -$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon_{jp}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon_{jq}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}} =$$

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{jp}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{jq}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}$$

$$-\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon_{jp}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon_{jq}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

As $\mathbb{t}_{\Upsilon j}(\alpha) \leq \mathbb{t}_{\Upsilon j}(\alpha)'$ and $\mathbb{f}_{\Upsilon j}(\alpha) \geq \mathbb{f}_{\Upsilon j}(\alpha)'$ $\forall\, j$, then

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{jp}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{jq}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}} =$$

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{jp}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{jq}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon_{j_p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon_{j_q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)=$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon_{j_p}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon_{j_q}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)$$

And

$$\Upsilon_j=\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{j_p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{j_q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}$$

$$+$$

$$\left(1-\left(1-\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{f}_{\Upsilon_{j_p}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{f}_{\Upsilon_{j_q}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)=$$

$$\left(1-\left(\bigotimes_{\substack{p,q=1\\ p\neq q}}^{N}\left(1-\left(\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{j_p}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\left(\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{j_q}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\frac{1}{N(N-1)}}\right)\right)^{\frac{1}{\ell+m}}+$$

$$\left(1-\left(1-\bigotimes_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(\begin{array}{c}1-\left(1-1-\left(\mathbb{e}xp^{-(\ell(-\ln(1-\mathbb{f}_{\Upsilon_{j\wp}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell}\\ \left(1-1-\left(\mathbb{e}xp^{-(m(-\ln(1-\mathbb{f}_{\Upsilon_{jq}}{}'(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)\right)^{m}\end{array}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}\right)=\Upsilon_j{}'$$

So,

$$IVIFAABM^{\ell,m}(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_N)=IVIFAABM^{\ell,m}(\Upsilon_1{}',\Upsilon_2{}',\ldots,\Upsilon_N{}') \tag{25}$$

*Theorem 4:* (*Commutativity*) Consider $\Upsilon_j=\left(\mathbb{t}_{\Upsilon j}(\alpha),\mathbb{f}_{\Upsilon j}(\alpha)\right)$ and $\Upsilon_j{}'=\left(\mathbb{t}_{\Upsilon j}(\alpha)',\mathbb{f}_{\Upsilon j}(\alpha)'\right)$; $(j=1,2,\ldots,N)$ be a assemblage of IVIF sets s.t $\mathbb{t}_{\Upsilon j}(\alpha)=\left[\mathbb{t}_{\Upsilon j}(\alpha)^{-},\mathbb{t}_{\Upsilon j}(\alpha)^{+}\right]$ and $\mathbb{f}_{\Upsilon j}(\alpha)=\left[\mathbb{f}_{\Upsilon j}(\alpha)^{-},\mathbb{f}_{\Upsilon j}(\alpha)^{+}\right]$ where $\mathbb{t}_{\Upsilon j}(\alpha)\leq\mathbb{t}_{\Upsilon j}(\alpha)'$ ; $\mathbb{f}_{\Upsilon j}(\alpha)\geq\mathbb{f}_{\Upsilon j}(\alpha)'$; $\forall\, j$, then

$$IVIFAABM^{\ell,m}(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_N)=IVIFAABM^{\ell,m}(\Upsilon_1{}',\Upsilon_2{}',\ldots,\Upsilon_N{}')$$

Where $(\Upsilon_1{}',\Upsilon_2{}',\ldots,\Upsilon_N{}')$ is any permutation of $(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_N)$.

*Proof*: As $(\Upsilon_1{}',\Upsilon_2{}',\ldots,\Upsilon_N{}')$ is any permutation of $(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_N)$, then

$$IVIFAABM^{\ell,m}(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_N)=([S_3^{-},S_3^{+}],[T_3^{-},T_3^{+}])$$

Where,

$$S_3^{-}=\mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp\left(-\left(\frac{1}{N(N-1)}\left(-ln\left(1-1-\left(\mathbb{e}xp\left(-\left(\sum_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon\wp}(\alpha)^{-}\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon q}(\alpha)^{-}\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)$$

$$S_3^+ = \mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp\left(-\left(\frac{1}{N\,(N-1)}\left(-ln\left(1-1-\left(\mathbb{e}xp\left(-\left(\sum_{\substack{\wp,q=1\\ \wp\neq q}}^{N}\left(ln\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_\wp}(\alpha)^+\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{t}_{\Upsilon_q}(\alpha)^+\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

$$T_3^- = 1-\mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp\left(-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_\wp}(\alpha)^-\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_q}(\alpha)^-\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

$$T_3^+ = 1-\mathbb{e}xp\left(-\left(\frac{1}{\ell+m}\left(-ln\left(1-\mathbb{e}xp\left(-\left(\frac{1}{N\,(N-1)}\left(-ln\left(\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_\wp}(\alpha)^+\right)^{\mathfrak{z}}\right)+m\left(-\ln\left(\mathbb{f}_{\Upsilon_q}(\alpha)^+\right)^{\mathfrak{z}}\right)\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}\right)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}$$

,

$$= IVIFAABM^{\ell,m}(\Upsilon_1{}', \Upsilon_2{}', \ldots, \Upsilon_N{}')$$

*Theorem 5:* (*Boundedness*) Consider $\Upsilon_j = \left(\mathbb{t}_{\Upsilon_j}(\alpha), \mathbb{f}_{\Upsilon_j}(\alpha)\right)$; ($j = 1,2,\ldots,N$) be an assemblage of IVIF sets, s.t $\mathbb{t}_{\Upsilon_j}(\alpha) = \left[\mathbb{t}_{\Upsilon_j}(\alpha)^-, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right]$ and $\mathbb{f}_{\Upsilon_j}(\alpha) = \left[\mathbb{f}_{\Upsilon_j}(\alpha)^-, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right]$ $\forall j$.
Suppose

$$\Upsilon^- = \left(\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha), \vee\, \mathbb{f}_{\Upsilon_j}(\alpha)\right) = \left(\left[\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)^-, \wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right], \left[\vee\, \mathbb{f}_{\Upsilon_j}(\alpha)^-, \vee\, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right]\right)$$

$$\Upsilon^+ = \left(\vee\, \mathbb{t}_{\Upsilon_j}(\alpha), \wedge\, \mathbb{f}_{\Upsilon_j}(\alpha)\right) = \left(\left[\vee\, \mathbb{t}_{\Upsilon_j}(\alpha)^-, \vee\, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right], \left[\wedge\, \mathbb{f}_{\Upsilon_j}(\alpha)^-, \wedge\, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right]\right)$$

Represents the lower and upper bound respectively. Then,

$$\Upsilon_j^- \leq IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N) \leq \Upsilon_j^+ \tag{26}$$

Which is the boundedness property.

*Proof*: As $\left[\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)^-, \wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right] \leq \left[\mathbb{t}_{\Upsilon_j}(\alpha)^-, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right] \leq \left[\vee\, \mathbb{t}_{\Upsilon_j}(\alpha)^-, \vee\, \mathbb{t}_{\Upsilon_j}(\alpha)^+\right]$; $\left[\wedge\, \mathbb{f}_{\Upsilon_j}(\alpha)^-, \wedge\, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right] \leq \left[\mathbb{f}_{\Upsilon_j}(\alpha)^-, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right] \leq \left[\vee\, \mathbb{f}_{\Upsilon_j}(\alpha)^-, \vee\, \mathbb{f}_{\Upsilon_j}(\alpha)^+\right]$ $\forall j$, then

$$\left(\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m} \leq \left( \mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}} \mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}} \right) \leq \left(\vee\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}$$

$$\otimes_{\substack{p,q=1\\p\neq q}}^{N} \left(1 - \mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}} \mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\frac{1}{N(N-1)}}$$
$$\leq \otimes_{\substack{p,q=1\\p\neq q}}^{N} \left(1 - \left(\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}\right)^{\frac{1}{N(N-1)}} = 1 - \left(\wedge\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}$$

$$\otimes_{\substack{p,q=1\\p\neq q}}^{N} \left(1 - \mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}} \mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\frac{1}{N(N-1)}}$$
$$\geq \otimes_{\substack{p,q=1\\p\neq q}}^{N} \left(1 - \left(\vee\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}\right)^{\frac{1}{N(N-1)}} = 1 - \left(\vee\, \mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}$$

And

$$\left(1-\otimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{t}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{t}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$\leq\left(1-\left(1-\left(\vee\,\mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}\right)\right)^{\frac{1}{\ell+m}}=\left(\vee\,\mathbb{t}_{\Upsilon_j}(\alpha)\right) \quad (27)$$

$$\left(1-\otimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\mathbb{e}xp^{-(\ell(-\ln(\mathbb{t}_{\Upsilon_{jp}}(\alpha)))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}},\mathbb{e}xp^{-(m(-\ln(\mathbb{t}_{\Upsilon_{jq}}(\alpha)^{-}))^{\mathfrak{z}})^{\frac{1}{\mathfrak{z}}}}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$\geq\left(1-\left(1-\left(\wedge\,\mathbb{t}_{\Upsilon_j}(\alpha)\right)^{\ell+m}\right)\right)^{\frac{1}{\ell+m}}=\left(\wedge\,\mathbb{t}_{\Upsilon_j}(\alpha)\right) \quad (28)$$

Also,

$$\left(1-\vee\,\mathbb{f}_{\Upsilon_j}(\alpha)\right)^{\ell+m}\leq\left(\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)$$

$$\leq\left(1-\wedge\,\mathbb{f}_{\Upsilon_j}(\alpha)\right)^{\ell+m}$$

$$\left(1-\vee\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_j}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}\leq\left(\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\right.$$

$$\left.\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)\leq\left(1-\wedge\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_j}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}$$

$$\otimes_{\substack{\mathcal{p},\mathcal{q}=1\\ \mathcal{p}\neq\mathcal{q}}}^{N}\left(1-\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{j\mathcal{p}}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{j\mathcal{q}}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)^{\frac{1}{N\,(N-1)}}$$

$$\geq \otimes_{\substack{\mathcal{p},\mathcal{q}=1\\ \mathcal{p}\neq\mathcal{q}}}^{N}\left(1-\left(1-\wedge\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}\right)^{\frac{1}{N\,(N-1)}}$$

$$=1-\left(1-\wedge\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}$$

And

$$\otimes_{\substack{\mathcal{p},\mathcal{q}=1\\ \mathcal{p}\neq\mathcal{q}}}^{N}\left(1-\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{j\mathcal{p}}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{j\mathcal{q}}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)^{\frac{1}{N\,(N-1)}}$$

$$\leq \otimes_{\substack{\mathcal{p},\mathcal{q}=1\\ \mathcal{p}\neq\mathcal{q}}}^{N}\left(1-\left(1-\vee\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}\right)^{\frac{1}{N\,(N-1)}}$$

$$=1-\left(1-\vee\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}$$

So,

$$1-\left(1-\otimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$\geq 1-\left(1-\left(1-\left(1-\wedge\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}}$$

$$=\wedge\left(1-\mathbb{e}xp^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)=\wedge\left(\mathbb{f}_{\Upsilon_{j}}(\alpha)\right) \quad (29)$$

And

$$1-\left(1-\otimes_{\substack{p,q=1\\p\neq q}}^{N}\left(1-\left(1-\mathbb{e}xp^{-\left(\ell\left(-\ln\left(\mathbb{f}_{\Upsilon_{jp}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{\ell}\left(1-\mathbb{e}xp^{-\left(m\left(-\ln\left(\mathbb{f}_{\Upsilon_{jq}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)^{m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}} \geq 1$$

$$-\left(1-\left(1-\left(1-\vee\left(1-\mathbb{exp}^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)\right)^{\ell+m}\right)^{\frac{1}{N(N-1)}}\right)^{\frac{1}{\ell+m}} \tag{30}$$

$$=\vee\left(1-\mathbb{exp}^{-\left(\left(-\ln\left(1-\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)\right)^{\mathfrak{z}}\right)^{\frac{1}{\mathfrak{z}}}}\right)=\vee\left(\mathbb{f}_{\Upsilon_{j}}(\alpha)\right)$$

Now, consider $\Upsilon = IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \left(\mathbb{t}_{\Upsilon}(\alpha), \mathbb{f}_{\Upsilon}(\alpha)\right)$ where $\mathbb{t}_{\Upsilon}(\alpha) = [\mathbb{t}_{\Upsilon}(\alpha)^-, \mathbb{t}_{\Upsilon}(\alpha)^+]$ and $\mathbb{f}_{\Upsilon}(\alpha) = [\mathbb{f}_{\Upsilon}(\alpha)^-, \mathbb{f}_{\Upsilon}(\alpha)^+]$. Then,

$$\mathcal{S}_{\Upsilon} = \left(\frac{1}{2}(\mathbb{t}_{\Upsilon 1}(\alpha)^- + \mathbb{t}_{\Upsilon 1}(\alpha)^+ - \mathbb{f}_{\Upsilon 1}(\alpha)^- - \mathbb{f}_{\Upsilon 1}(\alpha)^+)\right)$$
$$\geq \left(\frac{1}{2}\left((\wedge\, \mathbb{t}_{\Upsilon}(\alpha)^-) + (\wedge\, \mathbb{t}_{\Upsilon}(\alpha)^+) - (\vee\, \mathbb{f}_{\Upsilon 1}(\alpha)^-) - (\vee\, \mathbb{f}_{\Upsilon 1}(\alpha)^+)\right)\right) = \mathcal{S}_{\Upsilon^-} \tag{31}$$

$$\mathcal{S}_{\Upsilon} = \left(\frac{1}{2}(\mathbb{t}_{\Upsilon 1}(\alpha)^- + \mathbb{t}_{\Upsilon 1}(\alpha)^+ - \mathbb{f}_{\Upsilon 1}(\alpha)^- - \mathbb{f}_{\Upsilon 1}(\alpha)^+)\right)$$
$$\leq \left(\frac{1}{2}\left((\vee\, \mathbb{t}_{\Upsilon}(\alpha)^-) + (\vee\, \mathbb{t}_{\Upsilon}(\alpha)^+) - (\wedge\, \mathbb{f}_{\Upsilon 1}(\alpha)^-) - (\wedge\, \mathbb{f}_{\Upsilon 1}(\alpha)^+)\right)\right) = \mathcal{S}_{\Upsilon^+} \tag{32}$$

From this, consider the following cases:

i. If $\mathcal{S}_{\Upsilon} \leq \mathcal{S}_{\Upsilon^+}$ and $\mathcal{S}_{\Upsilon} \geq \mathcal{S}_{\Upsilon^-}$ then, form definition 3:

$$\mathcal{S}_{\Upsilon^-} \leq IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) \leq \mathcal{S}_{\Upsilon^+} \tag{33}$$

ii. If $\mathcal{S}_{\Upsilon} = \mathcal{S}_{\Upsilon^+}$, then from (30) and (32):

$$\mathbb{t}_{\Upsilon}(\alpha) = \left(\vee\, \mathbb{t}_{\Upsilon j}(\alpha)\right);\ \mathbb{f}_{\Upsilon}(\alpha) = \wedge\left(\mathbb{f}_{\Upsilon j}(\alpha)\right)$$

So,

$$\wp_{\Upsilon}(\alpha) = \mathbb{t}_{\Upsilon}(\alpha) + \mathbb{f}_{\Upsilon}(\alpha) = \left(\vee\, \mathbb{t}_{\Upsilon j}(\alpha)\right) + \left(\wedge\, \mathbb{f}_{\Upsilon j}(\alpha)\right) = \wp_{\Upsilon}(\alpha)^+$$

Then, form *Definition 3*:

$$IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) \leq \Upsilon^+ \tag{34}$$

iii. f $\mathcal{S}_{\Upsilon} = \mathcal{S}_{\Upsilon^-}$, by using (31) and (33):

$$\mathbb{t}_{\Upsilon}(\alpha) = \wedge\left(\mathbb{t}_{\Upsilon j}(\alpha)\right); \mathbb{f}_{\Upsilon}(\alpha) = \vee\left(\mathbb{f}_{\Upsilon j}(\alpha)\right)$$

So,

$$\wp_{\Upsilon}(\alpha) = \mathbb{t}_{\Upsilon}(\alpha) + \mathbb{f}_{\Upsilon}(\alpha) = \left(\wedge\left(\mathbb{t}_{\Upsilon j}(\alpha)\right)\right) + \left(\vee\left(\mathbb{f}_{\Upsilon j}(\alpha)\right)\right) = \wp_{\Upsilon}(\alpha)^- \tag{35}$$

Then, form *Definition 3*:

$$IVIFAABM^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \Upsilon^- \tag{36}$$

Which shows that (29) holds.
Now consider the following cases by putting the different values of parameters $\ell$ and $m$.

1) If $m \to 0$ then, form (12):

$$\lim_{m\to 0} \mathrm{IVIFAAB}^{\ell,m}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \lim_{m\to 0}\left(\frac{1}{N(N-1)}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N}\left(\Upsilon_p^{\ell} \otimes \Upsilon_q^{m}\right)\right)\right)^{\frac{1}{\ell+m}}$$

$$= \left(\frac{1}{N}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N} \Upsilon_p^{\ell}\right)\right)^{\frac{1}{\ell}} \tag{37}$$

Which is called IVIF AA Generalized Bonferroni Mean Operator.

2) If $\ell = 1$ and $m \to 0$ then, form (12):

$$\lim_{m\to 0} \mathrm{IVIFAAB}^{1,0}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \frac{1}{N}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N} \Upsilon_p^{1}\right) \tag{38}$$

Which is called IVIF AA Bonferroni Average Operator.

3) If $\ell = 2$ and $m \to 0$ then, form (12):

$$\lim_{m\to 0} \mathrm{IVIFAAB}^{2,0}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \left(\frac{1}{N}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N} \Upsilon_p^{2}\right)\right)^{\frac{1}{2}} \tag{39}$$

Which is called IVIF AA Bonferroni Square Mean Operator.

4) If $\ell = 1 = m$ then, form (12):

$$\lim_{m\to 0} \mathrm{IVIFAAB}^{1,1}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N) = \lim_{\ell,m\to 1}\left(\frac{1}{N(N-1)}\left(\oplus_{\substack{p,q=1\\ p\neq q}}^{N}\left(\Upsilon_p^{\ell} \otimes \Upsilon_q^{m}\right)\right)\right)^{\frac{1}{2}} \tag{40}$$

Which is called IVIF AA Bonferroni interrelated Square Mean Operator.

## 4. Assessment of the MCGDM problem based on the proposed Methodology:

In this section, we will discuss the MCGDM problem based on the proposed methodology to demonstrate its accuracy and effectiveness.

### *4.1 Algorithm*

To solve the MCGDM problem steps should be followed.

i. The formation of decision matrices by the group of experts and their opinions are stored in the form of matrices in terms of interval-valued intuitionistic fuzzy numbers (IVIFNs).
ii. Normalize the decision matrices if they contain cost-type data, otherwise, it does not require normalization.
iii. By utilizing the proposed operator, accumulate the decision matrices into a single matrix.
iv. Then, aggregate the accumulated decision matrix, using the proposed operators.
v. Evaluate the score values and then rank them in increasing order.

The pictorial representation is shown in Figure 1.

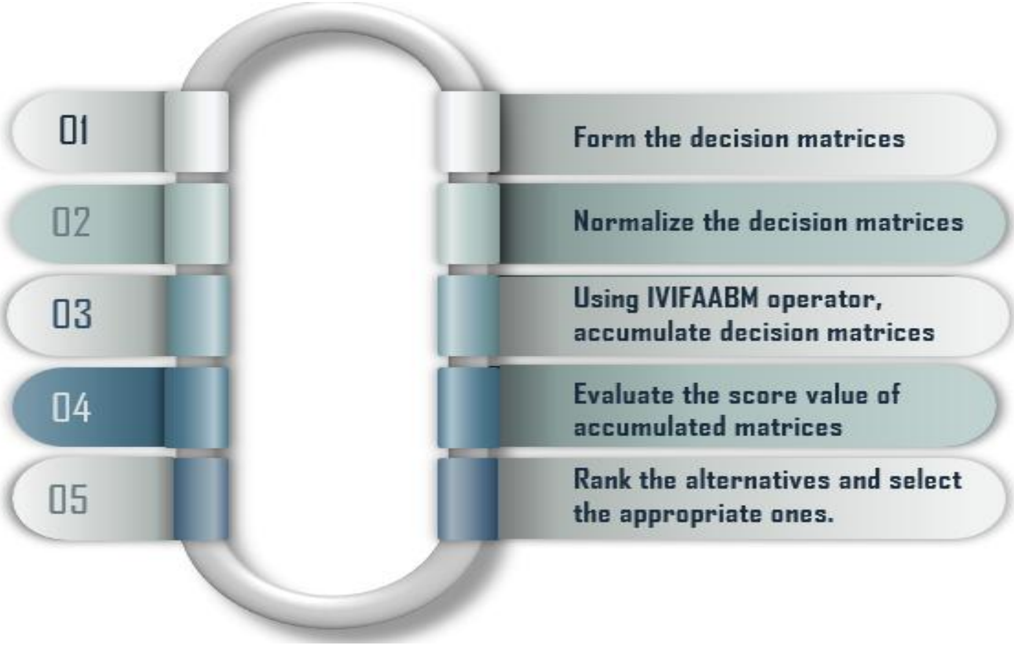


**Fig. 1.** Algorithm

### *4.2. Methodology*

To solve the MCGDM problem, the above steps are followed:

i. Consider $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N\}$ be a set of alternatives and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$ be a set of criteria's having weight vector $w = (w_1, w_2, \dots, w_N)^t$, s.t $\sum_{j=1}^{N} w_j = 1$ and $w_j > 0$, $\forall\, j$. By evaluating a problem in a proposed environment, each expertise is restricted for assigning values to each alternative in the form of IVIF set $\Upsilon_j = (\mathbb{t}_{\Upsilon j}, \mathbb{f}_{\Upsilon j})$ where $\mathbb{t}_{\Upsilon}$ and $\mathbb{f}_{\Upsilon}$ are in the form of closed interval and the value of each alternative corresponds to a parameter is recorded in the form of decision matrix $\mathcal{D} = (\Upsilon_{ij})_{N\times m}$ as

$$\mathcal{D} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_N \end{array} \begin{bmatrix} \mathcal{P}_1 & \mathcal{P}_2 & \dots & \mathcal{P}_m \\ \Upsilon_{11} & \Upsilon_{12} & \dots & \Upsilon_{1m} \\ \Upsilon_{21} & \Upsilon_{22} & \dots & \Upsilon_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{N1} & \Upsilon_{N2} & \dots & \Upsilon_{Nm} \end{bmatrix}$$

vi. The performance values don't need to be normalized if all the parameters are identical whereas in MCGDM, generally benefit and cost type parameters exist in which all cost type parameters transform into benefit type parameters by the following equation.

$$\mathfrak{r}_{ij} = \begin{cases} \bar{\Upsilon}_{ij}\ ;\ for\ cost\ type\ parameter \\ \Upsilon_{ij}\ ; for\ benefit\ type\ \ parameter \end{cases}$$

Where $\bar{\Upsilon}_{ij}$ is the complement of $\Upsilon_{ij}$ s.t $(\mathbb{f}_{\Upsilon j}, \mathbb{t}_{\Upsilon j})$.

vii. The IVIFAABM operator accumulates the decision-making matrices.

viii. Evaluate the score value of the accumulated matrix of step 3 by using (8).

$$\mathcal{S}_{\Upsilon} = \frac{1}{2}(\mathbb{t}_{\Upsilon 1}(\alpha)^- + \mathbb{t}_{\Upsilon 1}(\alpha)^+ - \mathbb{f}_{\Upsilon 1}(\alpha)^- - \mathbb{f}_{\Upsilon 1}(\alpha)^+)$$

ix. Rank the alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N\}$ in descending order and select the appropriate ones.

## 5. Selection of Industrial Robots

ISO 8373:2012 is related to the robots and robot operating devices in industrial and non-industrial environments [1]. A robot is an automated device with multiple programmable parameters that can carry out targeted activities according to the current situation without any interference from others. In this way, robots fill the void between automated machinery and human workers who can handle various tasks.

In current times, the selection of industrial robots is a challenging concern due to the rising sophistication and advancement of technology and capacities as these are constantly involved in manufacturing robots. Currently, industrial robots come in a wide range of variations having their distinct features, attributes, facilities, and specifications. Moreover, the manufacturing environment, production method cost, its design are some of the most critical demands that have an immediate effect on selecting the desired robot. So, to select the ideally suitable robot, the experts need to identify the lowest possible cost with the maximum desired and targeted output.

By considering this problem, we will discuss the MCGDM approach under the IVIF environment.

### *5.1 Numerical evaluation*

This example [40] is related to a selection of industrial robots for a specific industrial task. Consider the collection of robots $\mathcal{A}_i = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_5\}$ as a set of alternatives that are being selected based on some parameters $\mathcal{P}_i = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ including load carrying capacity (LCC), a memory carrying capacity (MCC), the velocity of the robot (VR), and accuracy of the robot (AR) respectively. LCC means the maximum load capacity that a robot can carry, and MCC means the set of instructions that is stored in the robot's memory for performing specific tasks. VR shows how fast the robot manipulates the specific task while AR is the measure of how closely the robot's end effectors are placed about the desired point which is normally expressed as the distance between the desired points and the center of each point where robots travels repeatedly throughout attempts. So, in preparation for a specific industrial task, the company's management team recruits specialists to identify and recommend the most suitable robot according to specified parameters. Each expertise is restricted to assigning the values in the form of an IVIF set corresponding to each alternative which is stored in the matrix form.

So, to achieve a specific task, we need to execute the MCGDM problem by following the defined algorithm.

The values assigned by each expertise are shown in Tables 1-3.

**Table 1**
Decision Matrix by Expert 1

| Alt. | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ |
|---|---|---|---|---|
| $\mathcal{A}_1$ | $\begin{pmatrix}[0.17,0.22],\\ [0.32,0.35]\end{pmatrix}$ | $\begin{pmatrix}[0.30,0.35],\\ [0.17,0.29]\end{pmatrix}$ | $\begin{pmatrix}[0.22,0.36],\\ [0.12,0.07]\end{pmatrix}$ | $\begin{pmatrix}[0.15,0.19],\\ [0.40,0.50]\end{pmatrix}$ |
| $\mathcal{A}_2$ | $\begin{pmatrix}[0.14,0.24],\\ [0.40,0.45]\end{pmatrix}$ | $\begin{pmatrix}[0.18,0.23],\\ [0.25,0.70]\end{pmatrix}$ | $\begin{pmatrix}[0.27,0.55],\\ [0.24,0.31]\end{pmatrix}$ | $\begin{pmatrix}[0.21,0.27],\\ [0.44,0.50]\end{pmatrix}$ |
| $\mathcal{A}_3$ | $\begin{pmatrix}[0.11,0.26],\\ [0.63,0.70]\end{pmatrix}$ | $\begin{pmatrix}[0.21,0.26],\\ [0.30,0.45]\end{pmatrix}$ | $\begin{pmatrix}[0.25,0.30],\\ [0.14,0.16]\end{pmatrix}$ | $\begin{pmatrix}[0.07,0.15],\\ [0.55,0.60]\end{pmatrix}$ |
| $\mathcal{A}_4$ | $\begin{pmatrix}[0.08,0.28],\\ [0.40,0.70]\end{pmatrix}$ | $\begin{pmatrix}[0.09,0.13],\\ [0.15,0.25]\end{pmatrix}$ | $\begin{pmatrix}[0.35,0.40],\\ [0.30,0.32]\end{pmatrix}$ | $\begin{pmatrix}[0.18,0.22],\\ [0.45,0.70]\end{pmatrix}$ |
| $\mathcal{A}_5$ | $\begin{pmatrix}[0.06,0.30],\\ [0.35,0.40]\end{pmatrix}$ | $\begin{pmatrix}[0.35,0.55],\\ [0.27,0.35]\end{pmatrix}$ | $\begin{pmatrix}[0.28,0.18],\\ [0.14,0.44]\end{pmatrix}$ | $\begin{pmatrix}[0.06,0.09],\\ [0.17,0.70]\end{pmatrix}$ |

**Table 2**
Decision Matrix by Expert 2

| Alt. | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ |
|---|---|---|---|---|
| $\mathcal{A}_1$ | $\begin{pmatrix}[0.13,0.16],\\ [0.15,0.19]\end{pmatrix}$ | $\begin{pmatrix}[0.07,0.10],\\ [0.41,0.45]\end{pmatrix}$ | $\begin{pmatrix}[0.15,0.19],\\ [0.27,0.35]\end{pmatrix}$ | $\begin{pmatrix}[0.02,0.03]\\ ,[0.41,0.50]\end{pmatrix}$ |
| $\mathcal{A}_2$ | $\begin{pmatrix}[0.15,0.20],\\ [0.30,0.50]\end{pmatrix}$ | $\begin{pmatrix}[0.15,0.16],\\ [0.31,0.35]\end{pmatrix}$ | $\begin{pmatrix}[0.18,0.23],\\ [0.24,0.31]\end{pmatrix}$ | $\begin{pmatrix}[0.04,0.06],\\ [0.15,0.20]\end{pmatrix}$ |
| $\mathcal{A}_3$ | $\begin{pmatrix}[0.17,0.19],\\ [0.40,0.50]\end{pmatrix}$ | $\begin{pmatrix}[0.11,0.19],\\ [0.08,0.09]\end{pmatrix}$ | $\begin{pmatrix}[0.21,0.27],\\ [0.15,0.19]\end{pmatrix}$ | $\begin{pmatrix}[0.06,0.09],\\ [0.11,0.20]\end{pmatrix}$ |
| $\mathcal{A}_4$ | $\begin{pmatrix}[0.19,0.27],\\ [0.35,0.55]\end{pmatrix}$ | $\begin{pmatrix}[0.03,0.09],\\ [0.13,0.16]\end{pmatrix}$ | $\begin{pmatrix}[0.24,0.31],\\ [0.21,0.26]\end{pmatrix}$ | $\begin{pmatrix}[0.08,0.12],\\ [0.18,0.20]\end{pmatrix}$ |
| $\mathcal{A}_5$ | $\begin{pmatrix}[0.25,0.30],\\ [0.40,0.42]\end{pmatrix}$ | $\begin{pmatrix}[0.11,0.21],\\ [0.02,0.08]\end{pmatrix}$ | $\begin{pmatrix}[0.27,0.35],\\ [0.18,0.23]\end{pmatrix}$ | $\begin{pmatrix}[0.10,0.15]\\ ,[0.16,0.30]\end{pmatrix}$ |

**Table 3**
Decision Matrix by Expert 3

| Alt. | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ |
|---|---|---|---|---|
| $\mathcal{A}_1$ | $\begin{pmatrix}[0.45,0.47],\\ [0.34,0.38]\end{pmatrix}$ | $\begin{pmatrix}[0.10,0.30],\\ [0.13,0.17]\end{pmatrix}$ | $\begin{pmatrix}[0.29,0.45],\\ [0.35,0.55]\end{pmatrix}$ | $\begin{pmatrix}[0.40,0.45],\\ [0.29,0.40]\end{pmatrix}$ |
| $\mathcal{A}_2$ | $\begin{pmatrix}[0.20,0.22],\\ [0.36,0.40]\end{pmatrix}$ | $\begin{pmatrix}[0.30,0.50],\\ [0.16,0.19]\end{pmatrix}$ | $\begin{pmatrix}[0.22,0.40],\\ [0.30,0.35]\end{pmatrix}$ | $\begin{pmatrix}[0.13,0.20],\\ [0.23,0.30]\end{pmatrix}$ |
| $\mathcal{A}_3$ | $\begin{pmatrix}[0.40,0.42],\\ [0.37,0.42]\end{pmatrix}$ | $\begin{pmatrix}[0.50,0.70],\\ [0.19,0.22]\end{pmatrix}$ | $\begin{pmatrix}[0.25,0.35],\\ [0.45,0.50]\end{pmatrix}$ | $\begin{pmatrix}[0.17,0.20],\\ [0.18,0.30]\end{pmatrix}$ |
| $\mathcal{A}_4$ | $\begin{pmatrix}[0.60,0.63],\\ [0.20,0.25]\end{pmatrix}$ | $\begin{pmatrix}[0.30,0.60],\\ [0.23,0.25]\end{pmatrix}$ | $\begin{pmatrix}[0.13,0.30],\\ [0.50,0.70]\end{pmatrix}$ | $\begin{pmatrix}[0.05,0.10],\\ [0.24,0.30]\end{pmatrix}$ |
| $\mathcal{A}_5$ | $\begin{pmatrix}[0.80,0.53],\\ [0.27,0.30]\end{pmatrix}$ | $\begin{pmatrix}[0.90,0.12],\\ [0.29,0.32]\end{pmatrix}$ | $\begin{pmatrix}[0.22,0.27],\\ [0.55,0.30]\end{pmatrix}$ | $\begin{pmatrix}[0.06,0.13],\\ [0.35,0.40]\end{pmatrix}$ |

As there is no cost type parameter, so skip step ii of the algorithm.

By step iii, accumulate the decision matrix by using the IVIFAABM operator which is displayed in Table 4.

**Table 4**
Collective preference of Experts by aggregating the above decision matrices by IVIFAABM AO

| Alt. | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ |
|---|---|---|---|---|
| $\mathcal{A}_1$ | $\begin{pmatrix}[0.2294,0.3773],\\ [0.9403,0.9564]\end{pmatrix}$ | $\begin{pmatrix}[0.1392,0.2628],\\ [0.9254,0.9570]\end{pmatrix}$ | $\begin{pmatrix}[0.2164,0.2820],\\ [0.9283,0.9697]\end{pmatrix}$ | $\begin{pmatrix}[0.1550,0.1452],\\ [0.9742,0.9870]\end{pmatrix}$ |
| $\mathcal{A}_2$ | $\begin{pmatrix}[0.1623,0.2661],\\ [0.9704,0.9888]\end{pmatrix}$ | $\begin{pmatrix}[0.2051,0.2867],\\ [0.9222,0.9898]\end{pmatrix}$ | $\begin{pmatrix}[0.2218,0.3345],\\ [0.9328,0.9608]\end{pmatrix}$ | $\begin{pmatrix}[0.1170,0.1362],\\ [0.9449,0.9637]\end{pmatrix}$ |
| $\mathcal{A}_3$ | $\begin{pmatrix}[0.2092,0.3699],\\ [0.9920,0.9969]\end{pmatrix}$ | $\begin{pmatrix}[0.2482,0.3328],\\ [0.8871,0.9374]\end{pmatrix}$ | $\begin{pmatrix}[0.2363,0.2768],\\ [0.9336,0.9526]\end{pmatrix}$ | $\begin{pmatrix}[0.0936,0.1246],\\ [0.9558,0.9740]\end{pmatrix}$ |
| $\mathcal{A}_4$ | $\begin{pmatrix}[0.2448,0.4828],\\ [0.9601,0.9958]\end{pmatrix}$ | $\begin{pmatrix}[0.1138,0.2205],\\ [0.8646,0.9078]\end{pmatrix}$ | $\begin{pmatrix}[0.2320,0.3054],\\ [0.9687,0.9906]\end{pmatrix}$ | $\begin{pmatrix}[0.0956,0.1285],\\ [0.9518,0.9850]\end{pmatrix}$ |
| $\mathcal{A}_5$ | $\begin{pmatrix}[0.3022,0.5627],\\ [0.9668,0.9757]\end{pmatrix}$ | $\begin{pmatrix}[0.3995,0.2619],\\ [0.8931,0.9321]\end{pmatrix}$ | $\begin{pmatrix}[0.2560,0.2472],\\ [0.9588,0.9627]\end{pmatrix}$ | $\begin{pmatrix}[0.0721,0.1138],\\ [0.9154,0.9925]\end{pmatrix}$ |

By using steps iv and v, evaluate the score values of each alternative and arrange them in descending order to determine the most suitable alternative, respectively which is shown in Table 5.

**Table 5**
Score Values of aggregated decision matrix

| $\mathcal{S}(\mathcal{A}_1)$ | $\mathcal{S}(\mathcal{A}_2)$ | $\mathcal{S}(\mathcal{A}_3)$ | $\mathcal{S}(\mathcal{A}_4)$ | $\mathcal{S}(\mathcal{A}_5)$ |
|---|---|---|---|---|
| $\mathbf{-0.3364}$ | $-0.4121$ | $-0.3814$ | $-0.4084$ | $-0.3321$ |

Based on the outcome presented above (Table 5), $\mathcal{A}_5$ is the most suitable choice (robot) for the specific industrial task. The graphical representation of the score value for each alternative (robot) is provided below in Figure 2.

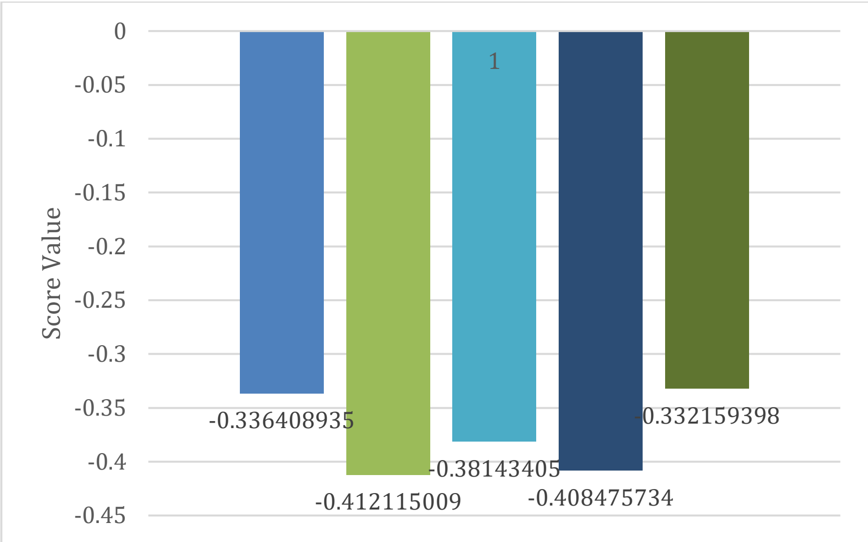


**Fig. 2.** Ranking of Alternatives (Robots)

*5.2 Impact of parameter*

To ensure the robustness, reliability, and impact of the proposed work it is essential to assess its stability through various parameters evaluation. So, by assigning the different values of parameters $\ell$ and $m$, we will check the stability of the proposed work and display the score values and ranking of each alternative (robot) in Table 6 and its pictorial representation is shown in Figure 3.

**Table 6**
Impact of parameter

| Parameters | Score Value | Ranking Values |
|---|---|---|
| $\boldsymbol{\ell = 1, m = 1}$ | $-0.3321, -0.3364, -0.3814, -0.4084, -0.4121$ | $\mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2$ |
| $\boldsymbol{\ell = 1, m = 2}$ | $-0.0730, -0.0808, -0.1347, -0.1468, -0.1707$ | $\mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4$ |
| $\boldsymbol{\ell = 1, m = 3}$ | $0.0473, 0.0411, 0.0105, 0.0058, -0.0164$ | $\mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_4$ |
| $\boldsymbol{\ell = 1, m = 10}$ | $0.0428, 0.0382, 0.0323, 0.0253, 0.0207$ | $\mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_4$ |
| $\boldsymbol{\ell = 2, m = 1}$ | $-0.0520, -0.0591, -0.1690, -0.1739, -0.1842$ | $\mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2$ |
| $\boldsymbol{\ell = 3, m = 1}$ | $0.1102, 0.0994, -0.0060, -0.0078, -0.0263$ | $\mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_2$ |
| $\boldsymbol{\ell = 10, m = 1}$ | $0.2420, 0.1686, 0.1428, 0.1282, 0.1008$ | $\mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_2$ |

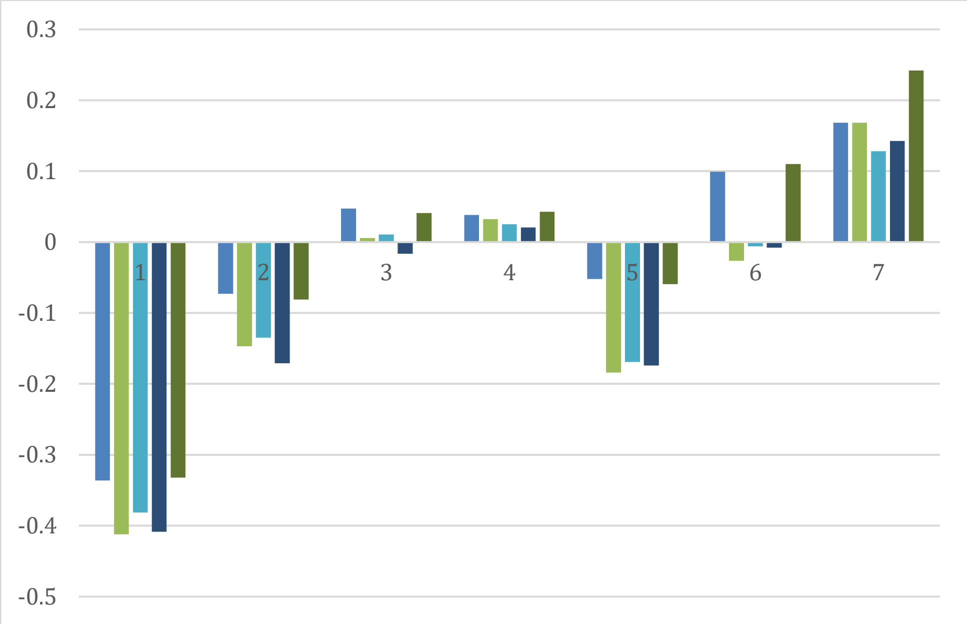


**Fig. 3.** Pictorial representation of Table 6

### *5.3 Comparison Analysis*

In this section, we have analyzed both the prior techniques and the newly proposed ones. This comparison highlights that the prior methods lack effectiveness and stability and cannot show the relationship between the multiple criteria values. Atanassov [12] introduced the concept of IVIF set which is unable to show the relationship between the multiple criteria. Then, Zhang [41] introduced the concept IVIF Frank AO, Wu *et al.,* [42] proposed the Dombi AO, Wang *et al.,* [43] discussed Einstein AO, Liu [19] defined the Hamacher AO, and AA AO was discussed by Senapati *et al.,* [16]. But all these operators do not give us precise and effective results. So, BM [38] was introduced which is a very effective operator and assists us in showing the relation between the multiple criteria. But to get more precise and efficient results, we proposed the hybrid structure of AA and BM in this paper which can be seen in Table 7.

**Table 7**
Comparison Analysis of the proposed and prior studies

| Operator | Relationship of Two Attributive Values | Relationship of Three Attributive Values | Relationship of Multiple Attributive Values | The parametric Method makes the Method Flexible | Hybrid Structure |
|---|---|---|---|---|---|
| Atanassov [12] | × | × | × | × | × |
| Zhang [41] | × | × | × | ✓ | × |
| Wu *et al.,* [42] | × | × | × | ✓ | × |
| Wang *et al.*, [43] | × | × | × | ✓ | × |
| Liu [19] | × | × | × | ✓ | × |
| Senapati [16] | × | × | × | ✓ | × |
| Xu *et al.,* [38] | ✓ | ✓ | ✓ | ✓ | × |
| AABM (Proposed) | ✓ | ✓ | ✓ | ✓ | ✓ |

## 6. Conclusion

The DM strategy involves formulating challenges and incidents within a mathematical framework. Various theories have been explored to address vague and uncertain data using a range of methodologies and tools. To tackle this type of information, we've introduced the AABM aggregate operator, a hybrid structure of AA and BM that gives more precise and effective results and assists the expertise in the selection of the best alternative. In this paper, we've implemented the efficient and well-known AABMAO and outlined the specific details as follows:

i. A novel approach to MCGDM by using the AABMAO for the IVIF set has been defined.
ii. The fundamental characteristics and attributes of the suggested theory and the suitable instrument for managing DM algorithms were determined.
iii. The adaptability and efficiency of the hybrid structure which is formed by combining the AA and BM theory and gives the most desirable outcomes has been illustrated.
iv. To demonstrate its superiority as the most versatile approach a comparative analysis with existing information has been discussed and a thorough examination of the proposed work's validity is defined.
v. A comprehensive analysis of the proposed approach's properties, efficiency, and flexibility compared to existing methods has been discussed.

### *6.1 Advantages*

The main advantage of AABMAO is that it offers flexibility which allows the researcher to adjust the parameters according to their requirement and get the most desirable result.

This operator captures the interrelation between the multiple criteria which gives a more realistic and efficient value.

### *6.2 Limitations and Future Directions*

Although the proposed approach provides us the efficiency and flexibility as compared to the prior approaches. However, it has some limitations s.t this approach is unable to deal with the problem where experts assign the values in a hesitancy environment [25], dual hesitancy environment [44], neutrosophic environment [45], and complex neutrosophic environment [46].

So, to consider this, in the future, we can extend this approach to complex intuitionistic environments, hesitant intuitionistic environments, neutrosophic environments, complex neutrosophic environments, Hybrid DEMATEL-EDAS based decision-making environments [47], DEA-Fuzzy ARAS Method for Prioritizing High-Performances [48] and application like vehicle's recycling facility location [49], smartphone preferences [50], sustainable emerging economics based on industry [51]. Moreover, these approaches have been tested to check their impact by addressing real-world challenges.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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