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Exchange Property in Double Edge Resolving Partition Sets and Its Use in City Development

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ABSTRACT

The exchange property in double-edge resolving partition sets is examined in this article, along with some real-world applications to city buildings. In graph theory, double-edge resolving sets are essential because they provide information on optimizing transportation and urban infrastructure. When utility units are switched out, the exchange property ensures the system is efficient and still works. City planners can create more adaptable and durable urban layouts by using this feature, guaranteeing that the best routes and shortest distances remain intact in various setups. We show how the exchange property in double-edge resolving partition sets can improve traffic management, emergency response systems, and overall urban planning through theoretical analysis and real-world case studies. The findings highlight the capability of graph-theoretical techniques in addressing complicated urban planning challenges, ultimately contributing to smarter, more sustainable town development. This study highlights the potential of advanced graph-theoretical concepts to address complex urban development challenges, contributing to the creation of smarter, more adaptive cities.

1. Introduction

Graph theory is a branch of mathematics, that researches the residences and programs of graphs, which are structures composed of vertices (or nodes) related by using edges (or hyperlinks). This subject has a wide range of programs across numerous disciplines, consisting of *PC* technological know-how, biology, social sciences, and engineering. In pc technology, the graph principle is essential to the layout and evaluation of algorithms, in particular in regions consisting of community layout, statistics enterprise, and computational biology. For example, in community design, graph theory helps optimize routes and enhance the efficiency of verbal exchange networks. In social sciences, it's miles used

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to version and analyze social networks, offering insights into the relationships and interactions inside a collection. Furthermore, in biology, graph ideas assist in understanding the complex relationships inside organic networks, which include food webs and neural networks. Its potential to version relationships and interactions makes graph theory a powerful device for fixing complicated problems and advancing understanding in numerous fields [1].

Network localization is a method for determining a node's precise location within a network. Precisely locating nodes in networks is a fundamental concept with numerous applications. It assists in locating the nearest printer, identifying a malfunctioning node, detecting network intrusions, pinpointing damaged devices, uncovering unauthorized connections, and tracking the position of a mobile robot when a computer sends a printing instruction within a facility. The process of network localization is indeed challenging, expensive, and time-consuming in the context of resolving sets, the study examines minimal resolving sets W_1 and W_2 for graph G. If each vertex in W_1 has a corresponding vertex in W_2, then the set obtained by replacing one vertex in W_1 with the corresponding vertex from W_2 is also a minimal resolving set. This concept is related to the EXP of resolving sets in the graph. Please refer to [2] for more details. This study considers finite, simple, and connected graphs. Additionally, all groups under consideration are finite. The graph's-exchange property is closely related to its set resolution property.

The unique representation of the relevant node with the chosen vertices identifies its position, enabling accurate placement to be established. This method is used to choose several nodes. We must select as few vertices as possible to make this process effective and energy-effective. The lowest set of selected vertices' cardinality is known as the locating number, and the set of selected vertices is known as the locating set (metric dimension). It is still unclear how to solve the NP-hard problem of determining the graph's location number; see [3, 4].

The concept of a resolving set, which is essential for distinguishing vertices based on their distances, has a rich history in graph theory. Slater introduced the idea, emphasizing the importance of having the smallest possible cardinality for a resolving set, known as the graph locating number [5]. He used examples such as Loran stations and sonar to illustrate the concept. Notably, Loran stations became outdated once GPS technology became widely available for commercial use. Harary and Melter also delved into this concept, coining the term metric dimension instead of location number to describe it [6]. Chartrand and colleagues referred to this concept as the metric basis, with the resolving set being the smallest subset of the metric basis [7]. In his monograph, Blumenthal thoroughly delved into the concept of resolving sets, examining their applications in distance geometry within the broader context of metric spaces [8].

The metric dimension finds numerous practical applications in our daily lives and serves as a rich source of inspiration for researchers. Some notable applications of the metric dimension include Pharmaceutical Chemistry: The metric dimension is used to determine similar patterns among various medications, aiding in drug discovery and development [9]. Combinatorial Optimization: Metric dimension plays a crucial role in combinatorial optimization problems, helping find the most efficient solutions in various fields [10, 11]. Robot Navigation: In robotics, the metric dimension is applied to assist robots in navigation and path planning, ensuring they can efficiently locate and reach their destinations [12,13]. Computer Networks: Metric dimension is employed in computer networks to enhance network efficiency and ensure data packets reach their intended destinations accurately [14]. Graph Theory: The metric dimension is valuable in the canonical labeling of graphs, simplifying graph-related analyses and algorithms [15]. Location Problems: It aids in solving location-related problems, such as determining the optimal allocation of facilities, services, or resources. Sonar and Coast Guard: The concept of metric dimensions practical applications in sonar technology and Coast Guard operations, facilitating precise location determination [16]. Image Processing: In image processing, metric dimension is used for tasks like object recognition, where it helps identify and locate objects in images.

Weighing Problems: Metric dimensions are applied in addressing weighing problems, such as measuring unknown weights by comparing them to known weights [17]. Cryptography: The metric dimension is used in coding and decoding systems, as seen in the coding and decoding of the Mastermind game [18-22].

The metric dimension of graphs has been the subject of extensive research over the past four decades, leading to insights and findings in various areas. Some notable contributions in this field include Honeycomb Networks: Research by [23] focused on the metric dimension of honeycomb networks, exploring the specific characteristics of these structures. Silicate Stars: The metric dimension of silicate star graphs was discussed in [24], providing insights into the resolving sets of such networks. Cellulose Networks: The upper bounds of the metric dimension for cellulose networks were determined in a study by [25], shedding light on the metric properties of these networks. Symmetric Graphs: The metric dimension of symmetric graphs was investigated using rooted product techniques in the work by [26], offering a comprehensive understanding of their resolvability. Crystal Cubic Carbon Structure: The metric dimension of the crystal cubic carbon structure was analyzed in [29], contributing to the understanding of the metric dimension in complex molecular structures. Convex Polytopes Graph: Double edge resolving set and exchange property discussed in[27] and [29]discussed the metric dimension of the graph representing convex polytopes, uncovering the resolvability aspects of these mathematical constructs. Cayley Graphs Barycentric Subdivision: Research by [30] delved into the edge metric dimension of Cayley graphs with barycentric subdivisions, extending the application of metric dimension concepts to a diverse range of graphs. Recently a new resolvability parameter was introduced by[31] and the use of resolvability parameters in anti-malaria drugs is discussed by [32]. The concept of metric dimension has proven to be a valuable tool for solving various challenging problems across different domains. For insights into the resolvability parameters of chemical structures, researchers can refer to works like [33-35], which provide a deeper understanding of how metric dimension concepts are applied in the field of chemistry. These studies collectively demonstrate the broad applicability and significance of the metric dimension in diverse graph structures and real-world scenarios, facilitating problem-solving and analytical insights.

For additional insights, we recommend referring to the following sources: [36-40]. This article also relies on fundamental mathematical definitions related to distance, resolving sets, and metric dimensions. In this work, some notions are used like *PRS* for partition resolving set and *EXP* for exchange property.

2. Hexagonal Nano-network

Due to its advantages over other lattice patterns, hexagonal networks are used in numerous scientific domains. The ambiguity of a square network is removed by the simpler and more symmetric close neighborhood of a hexagonal network. The rectangular network might not be the best choice when connections, surroundings, or paths are important. According to recent studies on digital image processing, a hexagonal network outperforms a square network in terms of performance[41-44]. Hexagons can be used for observation, research, and simulation in ecology, with natural displays having a specific benefit, according to Birch [45]. In particular, for cartography, several academics recommend using hexagonal networks because of their ability to achieve fine resolution by making use of the division of larger cells into smaller ones [46, 47]. Using a hexagonal network is indisputable since it offers greater advantages than other conventional networks that make use of square or rectangular arrays.

3. Double Edge Resolving Set of Hexagonal nanosheet

The structures have double resolving sets or more resolving sets and have *EXP* that are very important to find the best locations for projects. In this article, we add an application of *EXP* that is very important.

3.1 Construction of hexagonal Nanosheet

In Figure 1, vertices and edges are color-coded for distinction. The edges that ended on vertices of degrees 2 and 3 are marked in red and the edges that ended at degree 2 vertices are colored blue, while edges that ended with degree 3 are shown in black. Vertices with a degree of two are highlighted in green. The vertices with dual colors represent the points in the *PRS*. Specifically, $a_{1,1}$ and $a_{1,2h+1}$ are shown in red and green due to their degree two and belonging to *PRS*. These vertices are part of the *PRS*. In this context, v and h refer to the vertical and horizontal numbers of C_6, where h and v are greater than or equal to 1, and they are integers. $2h+v+3$ are the vertices of degree 2 while $5h+4v-3-(2h+v)$ are pf degree 3. $|V(HNS_{h,v}| = (2h+1)(v+1)$ is the order and $|E(HNS_{h,v}| = (2h+1)(v+1)+1$. is the size of the $HNS_{h,v}$.

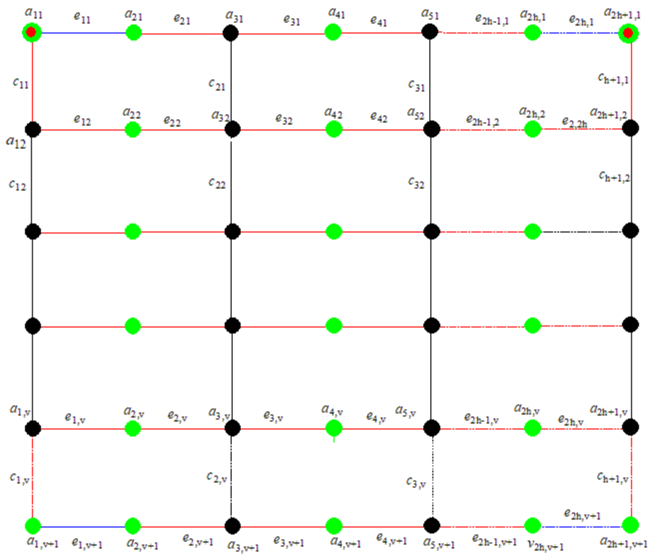

Fig. 1. Generalize Nanosheet derived from Hexagone

Two parameters v, h and two index s, t are used for labling, $1 \leq s \leq v$ and t change 2 time with h. Moreover, Figure 1 shows the labeling defined above for edge and vertex sets and used in our main results. Moreover, 1 shows the labeling defined above in edge and vertex sets and used in our main results. The edges are labeled as $a_{s,t}a_{s,t+1} = c_{s,t}$. The vertices and edges sets of the hexagonal nanosheet are

$$V(HNS) = \{a_{s,t}; 1 \leq s \leq v, 1 \leq t \leq 2h+1\}$$
$$E(HNS) = \{a_{s,t}a_{s,t+1}, a_{s,t}a_{s+1t}; 1 \leq s \leq v,\ 1 \leq t \leq 2h+1\}$$

Theorem 3.1. We consider a hexagonal nanosheet represented as $HNS_{h,v}$ with both h and v greater than or equal to 1, it can be confirmed that there exist double-edge partition sets of the cardinality of 3.

Proof. To demonstrate the nanosheet double edge PRS, two sets, W_1 and W_2, are defined. The next step is to demonstrate that both W_1 and W_2 are PRSs for the nanosheet. Theorem 4.2 contains the proof of W_1, while Theorem 5.1 contains the proof of W_2. ■

Theorem 3.2. In the case of a hexagonal nanosheet denoted as $HNS_{h,v}$ with $h, v \geq 1$, it can be established that W_1 forms an edge PRS with cardinality of 3.

Proof. To demonstrate that both edge-PRSs of $HNS_{h,v}$ have a minimum cardinality of 3, we will follow the definition of an edge-PRS. Let $W_{1p1} = a_{1,1}$, $W_{1p2} = a_{2h+1,1}$ and $W_{1p3} = v(HNS) \backslash \{a_{1,1}, a_{2h+1,1}\}$ then $W_1 = \{W_{1p1}, W_{1p2}, W_{1p3}\}$ is a partition set. Now we want to prove that the W_1 is an edge partition resolving. In the next theorem, we will show that $W_2 = \{W_{2p1}, W_{2p2}, W_{2p3}\}$ also serves as a edge PRS for the $HNS_{h,v}$ for $h, v = 1$. ■

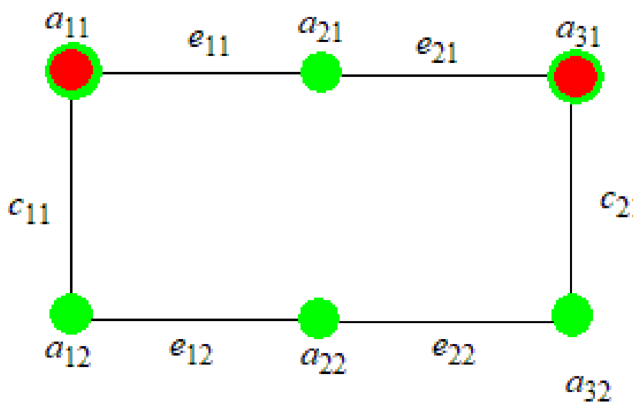

Fig. 2. Hexagon

The representation of $e_{s,t}$ for one hexagon using the edge-PRS $W_1 = \{W_{1p1}, W_{1p2}, W_{1p3}\}$ is in Table 1.

Table 1
Representation of edges of Fig. 2

Edges	$e_{1,1}$	$e_{1,2}$	$e_{2,1}$	$e_{2,2}$	$c_{1,1}$	$c_{1,2}$
$r(.\|R)$	**(0,1,0)**	**(1,0,0)**	**(1,2,0)**	**(2,1,0)**	**(0,2,0)**	**(2,0,0)**

Table 1 shows the unique representation of all edges of Figure 2 for W_1 so W_1 is edge-PRS. Now check the representation of Figure 3 in Table 2

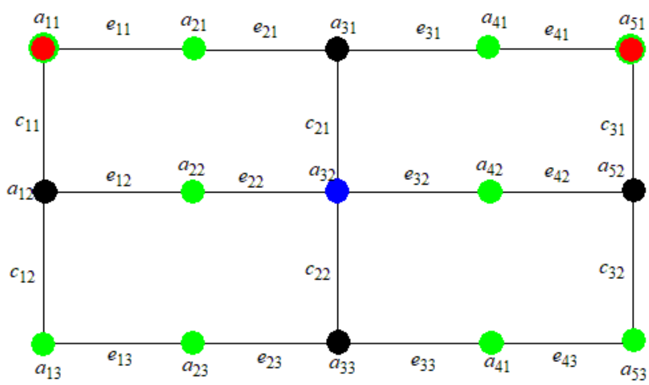

Fig. 3. Hexagon

Table 2
Edges of Figure 3 are represented

Edges	$e_{1,1}$	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$	$e_{2,1}$	$e_{2,2}$
$r(.\|R)$	(0,3,0)	(1,2,0)	(2,1,0)	(3,0,0)	(1,4,0)	(2,3,0)
Edges	$e_{2,3}$	$e_{2,4}$	$e_{3,1}$	$e_{3,2}$	$e_{3,3}$	$e_{3,4}$
$r(.\|R)$	(3,2,0)	(4,1,0)	(2,5,0)	(3,4,0)	(4,3,0)	(5,2,0)
Edges	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$
$r(.\|R)$	(1,4,0)	(2,2,0)	(4,1,0)	(1,5,0)	(3,3,0)	(5,1,0)

Table 2 also shows the unique representation. Now check the generalized formulas for W_1.

Generalized distances

The general formulas for the distances between all vertices within the hexagonal nanosheet illustrate that the metric dimension is two because each pair of vertices has distinct distances between them. Let $d(a_{s,t}, W_{1p1}) = \alpha_1, d(a_{s,t}, W_{1p2}) = \alpha_2, d(a_{s,t}, W_{1p3}) = \alpha_3$ and $r(a_{s,t}|W_1 = (\alpha_1, \alpha_2, \alpha_3)$

$$\alpha_1 = \{s+t-2 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{1}$$
$$\alpha_2 = \{2h+s-t-1 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{2}$$
$$\alpha_3 = \{0 \quad \text{for all edges.} \tag{3}$$

Let $d(c_{s,t}, W_{1p1}) = \alpha_4, d(c_{s,t}, W_{1p2}) = \alpha_5, d(c_{s,t}, W_{1p3}) = \alpha_6$ and $r(c_{s,t}|R) = (\alpha_4, \alpha_5, \alpha_6)$

$$\alpha_4 = \{2s+t-3 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{4}$$
$$\alpha_5 = \{2h+s-2t+1 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{5}$$
$$\alpha_6 = \{0 \text{ for all edges.} \tag{6}$$

Let ξ and η are two any random nodes on hexagonal nanosheet $HNS_{h,v}$. Let $R = \{a_{1,1}, W_{1p2}\}$.

Case I: When $p = e_{s,t}$ and $q = e_{s',t'}$, then further three cases arise.

Case 1: if $s = s', t \neq t'$ and WLOG we can say that $t < t'$ then $d(\xi, a_{1,1}) \neq d(\eta, a_{1,1})$ Because $d(\xi, a_{1,1}) = d(\eta, a_{1,1}) + n$ where $= t' - t$ so $r(\xi|W_1) \neq r(\eta|W_1)$.

Case 2: if $s \neq s', t = t'$ and WLOG we can say that $s < s'$ then $d(\xi, a_{1,1}) \neq d(\eta, a_{1,1})$ Because $d(\xi, a_{1,1}) = d(\eta, a_{1,1}) + m$ where $m = 2(s' - s)$ so $r(\xi|W_1) \neq r(\eta|W_1)$.

Case 3: if $s \neq s', t \neq t'$ and WLOG we can say that $s < s', t < t'$ then $d(\xi, W_1) \neq d(\eta, W_1)$ Because $d(\xi, W_1) = d(\eta, W_1) + (m+n)$ so $r(\xi|W_1) \neq r(\eta|W_1)$.

Based on the previously presented representations and discussions, it is evident that a unique representation is provided for all edges, satisfying the conditions of a PRS, which demonstrates that $|R| = 3$.

Conversely For $epd(HNS_{h,v}) \geq 3$

$$\sim epd(HNS_{h,v}) < 3$$
$$\implies epd(HNS_{h,v}) = 1, 2.$$

The edge partition dimension is minimum 2 only for a path graph, this is not a path graph so 2 is not possible. Hence prove that the edge partition dimension of hexagonal nanosheet is 3.
In the next subsection we want to prove that the nanosheet has another edge partition re4solvinng set of the cardinality 3.

3.2 W_2 is also Edge Partition Resolving Set of Hexagonal Nanosheet

The construction of Figure 4 is same as Figure 1 but the representation of edge partition resolving set is different.

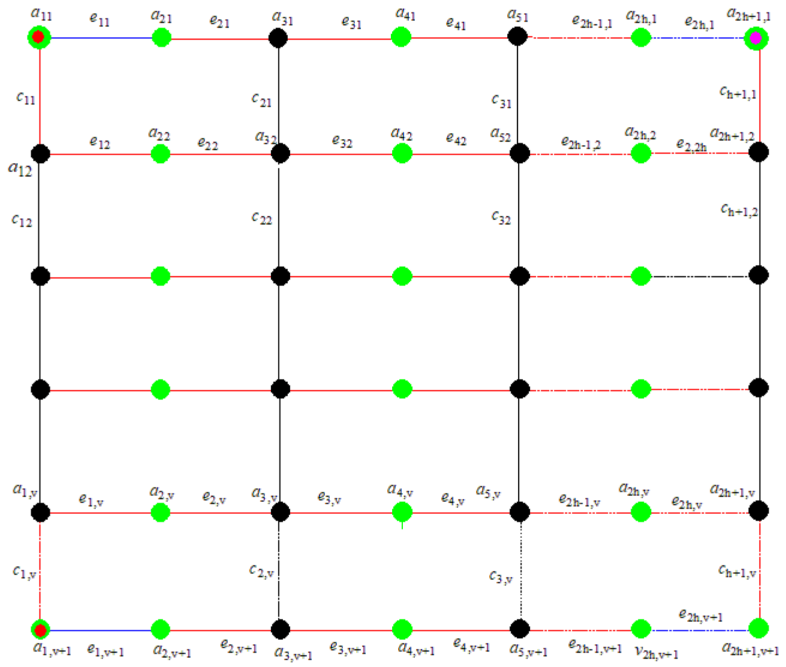

Fig. 4. Generalized hexagonal Nanosheet

Theorem 3.3. In the case of a hexagonal nanosheet, denoted as $HNS_{h,v}$ with $h, v \geq 1$, it can be shown that W_2 is also serves as an edge PRS with the cardinality of 3.

Proof. The definition of an edge-PRS will be used to demonstrate that the edge-PRS of $HNS_{h,v}$ has a minimum caredinality of 3. Let $W_{2p1} = a_{1,1}, W_{2p2} = a_{1,v+1}$ and $W_{2p3} = v(HNS) \backslash \{a_{1,1}, a_{1,v+1}\}$ then $W_1 = \{W_{2p1}, W_{2p2}, W_{2p3}\}$ is a partition set. Now we want to prove that the W_1 is an edge partition resolving for the $HNS_{h,v}$ nanosheet. For $h, v = 1$, the unique representation of every edge of $h, v = 1$ is shown below in the table. ■

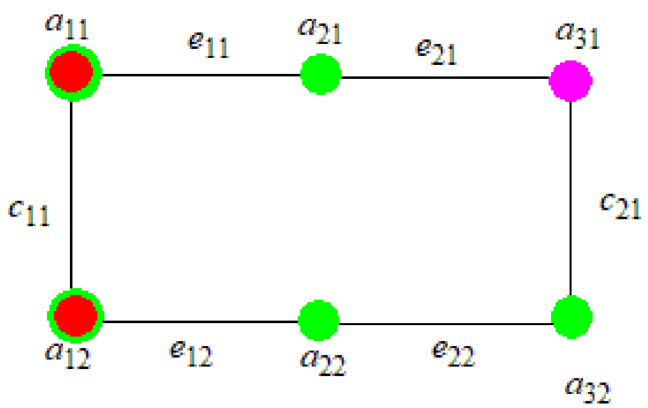

Fig. 5. Hexagon

The representation of $e_{s,t}$ for $h = 1 = v$ present in Table 3

Table 3
Edge representation of Figure 2

Edges	$e_{1,1}$	$e_{1,2}$	$e_{2,1}$	$2_{2,2}$	$c_{1,1}$	$c_{1,2}$
$r(.\|R)$	**(0,1,0)**	**(1,2,0)**	**(1,0,0)**	**(2,1,0)**	**(0,0,0)**	**(2,2,0)**

Table 3 shows the unique representation of all edges of Figure 5 for W_2 so W_2 is edge-PRS. Now check the representation of Figure 6 in Table 4

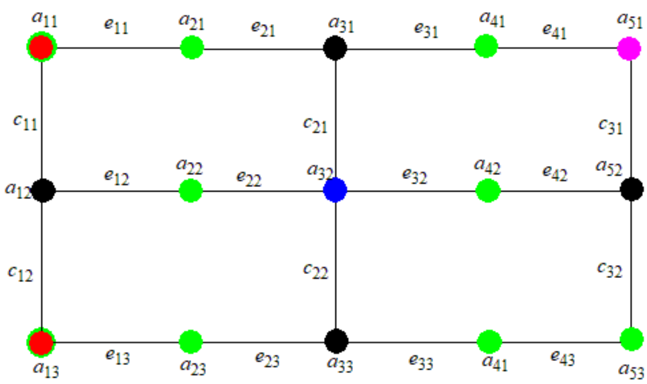

Fig. 6. Hexagon

The representation of $e_{s,t}$ for $h = 2 = v$ present in table 4

Table 4
Representation of vertices of Figure 6

Edges	$e_{1,1}$	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$	$e_{2,1}$	$e_{2,2}$
$r(.\|R)$	**(0,2,0)**	**(1,3,0)**	**(2,4,0)**	**(3,5,0)**	**(1,1,0)**	**(2,2,0)**
Edges	$e_{2,3}$	$e_{2,4}$	$e_{3,1}$	$e_{3,2}$	$e_{3,3}$	$e_{3,4}$
$r(.\|R)$	**(3,3,0)**	**(4,4,0)**	**(2,0,0)**	**(3,1,0)**	**(4,2,0)**	**(5,3,0)**
Edges	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$
$r(.\|R)$	**(0,1,0)**	**(2,3,0)**	**(4,5,0)**	**(1,0,0)**	**(3,2,0)**	**(5,4,0)**

Generalized Distances

The generalized distance formulas for all edges of the hexagonal nanosheet indicate that the metric dimension is two because all distances are distinct. Let $d(e_{s,t}, W_{2p1}) = \alpha_1, d(e_{s,t}, W_{2p1}) = \alpha_2, d(e_{s,t}, W_{2p3}) = \alpha_3$ and $r(e_{s,t}|W_2 = (\alpha_1, \alpha_2, \alpha_3)$

$$\alpha_1 = \{s + t - 2 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{7}$$
$$\alpha_2 = \{h + s - t \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{8}$$
$$\alpha_3 = \{0 \text{ for all edges.} \tag{9}$$

Let $d(c_{s,t}, W_{2p1}) = \alpha_4, d(c_{s,t}, W_{2p2}) = \alpha_5, d(c_{s,t}, W_{2p3}) = \alpha_6$ and $r(c_{s,t}|R) = (\alpha_4, \alpha_5, \alpha_6)$

$$\alpha_4 = \{2s+t-3 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{10}$$
$$\alpha_5 = \{h+2s-t-2 \text{ for } 1 \leq s \leq h, \text{ and } 1 \leq t \leq v, \tag{11}$$
$$\alpha_6 = \{0 \text{ for all edges.} \tag{12}$$

Let ξ and η are two arbitrary edges on nanosheet $HNS_{h,v}$. Let $R = \{a_{1,1}, a_{1,v+1}\}$ and Let WLOG denotes without loss of generality.

Case I: When $p = e_{s,t}$ and $q = e_{s',t'}$, then further three cases arise.

Case 1: if $s = s', t \neq t'$ and WLOG we can say that $t < t'$ then $d(\xi, a_{1,v+1}) \neq d(\eta, a_{1,v+1})$ Because $d(\xi, a_{1,v+1}) = d(\eta, a_{1,v+1}) + n$ where $n = t' - t$ so $r(\xi|W_2) \neq r(\eta|W_2)$.

Case 2: if $s \neq s', t = t'$ and WLOG we can say that $s < s'$ then $d(\xi, a_{1,v+1}) \neq d(\eta, a_{1,v+1})$ Because $d(\xi, a_{1,v+1}) = d(\eta, a_{1,v+1}) + m$ where $m = 2(s' - s)$ so $r(\xi|W_2) \neq r(\eta|W_2)$.

Case 3: if $s \neq s', t \neq t'$ and WLOG we can say that $s < s', t < t'$ then $d(\xi, W_2) \neq d(\eta, W_2)$ Because $d(\xi, W_2) = d(\eta, W_2) + (m+n)$ so $r(\xi|W_2) \neq r(\eta|W_2)$.

Based on the preceding discussion and the representations provided, it is evident that a unique representation is achieved for all vertices, satisfying the conditions of an edge-PRS, thereby demonstrating that the metric dimension is two. $|R| = 3$.

Conversely For $epd(HNS_{h,v}) \geq 3$

$$\sim epd(HNS_{h,v}) < 3$$
$$\implies epd(HNS_{h,v}) = 1, 2.$$

The edge partition dimension is a minimum of 2 and that is for the path graph, this is not a path graph so its partition is not 2. Hence edge partition dimension is also 3 for with W_2. From the Theorems,3.1 and 3.3 shows that W_1 and W_2 are the edge PRSs of nanosheet with cardinality 3.

Now, we aim to demonstrate that the EXP holds in both W_1 and W_2 for the hexagonal nanosheet $HNS_{h,v}$. In vector spaces, a basis is a set of components that uniquely determines each vector through a linear combination. This concept includes the EXP, which holds for vector space bases. In a finite graph, minimal PRSs' vertices serve a similar purpose, uniquely identifying each vertex. Essentially, these PRSs can be likened to bases in finite-dimensional vector spaces. However, it's essential to recognize that, unlike linear bases in vector spaces, minimal PRSs may not always exhibit the EXP. The literature in this field explores the presence or absence of the EXP in various graph types. For instance, the EXP is true for PRSs in trees, but for wheel graphs W_n, it is false for $t \geq 8$.

Theorem 3.4. Let $HNS_{h,v}$, be a hexagonal nanosheet with $h, v \geq 1$ then EXP for edge partition resolving sets hold for this structure.

Proof. To establish the EXP for $HNS_{h,v}$, we commence with the definition of EXP. Let $u = a_{2h+1,1} = W_{1p2}$, where $u \in W_1$, and $v = a_{1,v+1} = W_{2p2}$, with $v \in W_2$. Now, consider the set $(W_1 \backslash u) \cup v = K$. The objective is to demonstrate that K also functions as a minimal PRS for $HNS_{h,v}$. Since we know that $W_1 = \{W_{1p1}, W_{1p2}, W_{1p3}\}$ and $W_2 = \{W_{2p1}, W_{2p2}, W_{2p3}\}$ it follows that $(W_1 \backslash u) = \{W_{1p1}, W_{1p3}\} \cup v = \{W_{1p1}, W_{2p2}, W_{1p3}\} = W_2$ because $W_{2p1} = W_{1p1}$ and W_{1p3} has same representation as W_{2p3}. Furthermore, Theorem3.3 has established that W_2 is a minimal Edge-PRS. Hence, the EXP holds for $HNS_{h,v}$ since the points W_{1p1} and W_{2p1} are common to both W_1 and W_2. This interchange of u and v between W_1 and W_2 confirms the EXP for the nanosheet. ■

4. Application of Exchange property

Problem statement:

- A developer is starting to build a new city, and he wants to know where and how many utility units are constructed. He will need to provide water, power, and other necessities to every road.
- He wants to give every road in the city a special code.
- He emphasizes the need for this coding scheme to maintain its uniqueness and accurately represent the shortest distance, even if any utility unit is replaced by another location. The code should effectively denote the shortest distance from the road to any utility unit.

A mathematician is consulted by the developer in an attempt to find a solution.

Solution:

- The mathematician turns the city map into a mathematical graph, where the edges represent the streets and the vertices represent the houses.
- He then applies the definition of double-edge resolving sets.
- Based on this definition of edge RS, he creates a set W_1 of minimum utility units and calculates the distances from each street.
- W_1 gives unique representation with all streets or roads.
- Next, he checks another set of utility units, W_2, for replacement.
- The other set W_2 also has a unique representation with all streets.
- As a result of his work, the developer can build two utility units at the corners of the city. These locations meet the requirements for the coding system to maintain its uniqueness and accurately reflect the shortest distance, satisfying his criteria. This remains true even if one utility unit replaces the position of another.

Example:

The map of the city is shown in Figure 7 that a mathematician converts into Figure 8

Fig. 7. City structure

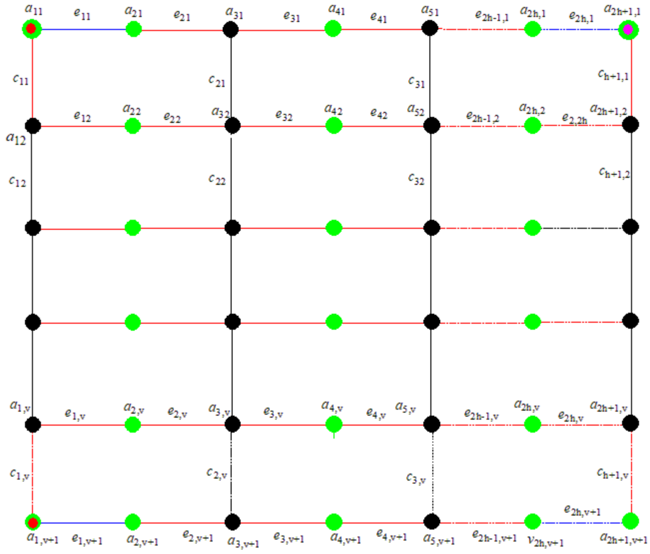

Fig. 8. City structure

If two individuals placed orders for customer services from e_{42} and e_{54}. Upon receiving these orders, the service provider checks the accompanying codes. The codes provided are $(4,7)$ and $(7,4)$. To efficiently provide services, the service provider uses these codes to identify the nearest units to the respective houses and streets and proceeds to offer the required services. For example, for e_{42}, unit 1 is chosen as it is the closest, while for e_{54}, unit 2 is selected because it is closed see 8.

5. Conclusion

We study hexagonal nanosheets in this work and compute two minimal edge-PRSs, each with cardinality two. We also confirm that these sets have the EXP. The EXP holds in $HNS_{h,v}$ according to our findings. More specifically, a common point in both W_1 and W_2 is the vertex $a_{1,1}$. As a result, W_1 and W_2 are able to trade the points u and v. Since W_1 and W_2 are minimal edge PRSs, we can conclude that the hexagonal nanosheet has an edge metric dimension of 2. We also investigate the utilization of the double edge-PRS in this setting, emphasizing its potential applications and practical importance. This investigation offers a solid foundation for upcoming studies and applications in the realm of nanotechnology and sheds light on how these ideas might be used to improve our comprehension and analysis of hexagonal nanosheets.

Data Availability

All the data supporting the results are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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