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Socio-Economic Development in Chinese Educational Systems Based on Robust Aggregation Operators for Managing Complex q-Rung Orthopair Fuzzy Uncertainty and Their Applications

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ABSTRACT

Sustainable development of the Chinese education sector demands a comprehensive investment strategy that integrates public-private partnerships, early childhood education, community engagement, teacher capacity building, curriculum relevance, equitable resource distribution, and continuous quality monitoring. Such an inclusive framework strengthens human capital and supports long-term economic growth. To effectively manage uncertainty and vagueness in multi-attribute group decision-making (MAGDM) problems arising from such investment evaluations, this study aims to develop novel aggregation operators within the complex q-rung orthopair fuzzy (Cq-ROF) environment. For this purpose, newly defined algebraic complex operational laws are introduced to construct the complex q-rung orthopair fuzzy weighted arithmetic mean (Cq-ROFWAM), ordered weighted arithmetic mean (Cq-ROFOWAM), weighted geometric mean (Cq-ROFWGM), and ordered weighted geometric mean (Cq-ROFOWGM) operators. These operators provide a flexible and robust framework for aggregating uncertain and complex evaluation information while effectively capturing decision-makers' preferences. Based on these operators, a novel MAGDM model is developed to support investment assessment in the Chinese education sector. An illustrative case study demonstrates the applicability and practicality of the proposed approach. Furthermore, comparative and numerical analyses verify that the proposed model produces stable and consistent alternative rankings, accurately identifies the optimal investment strategy, and offers reliable decision support. The main contributions of this work include the introduction of new Cq-ROF aggregation operators based on algebraic complex operational laws, the development of a comprehensive MAGDM framework, and the provision of a flexible and reliable decision-making methodology that outperforms existing approaches in handling uncertain information.

1. Introduction

In recent years, multiple-attribute decision-making (MADM) and multiple-attribute group decision-making (MAGDM) methods have attracted considerable attention because of their ability to solve complex decision problems involving multiple and often conflicting criteria [1]. Recently, several stud-

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ies have focused on extending these approaches to model uncertainty in healthcare, COVID-19, and other complex decision-making applications [2]. Hybrid MAGDM approaches have also demonstrated remarkable effectiveness in evaluating sustainable enterprises and pharmaceutical industries under uncertain environments [3]. Education is widely recognized as one of the most important sectors for the social and economic development of a nation. Investment in education is generally regarded as an investment in human capital because it enhances knowledge, skills, and innovation. Such investments generate long-term benefits for individuals as well as society by improving productivity, strengthening social cohesion, and promoting sustainable economic growth. With the rapid development of global economies, industrial expansion has significantly accelerated economic growth. However, this rapid development has also raised several concerns regarding environmental protection, ecological sustainability, and social security. Therefore, achieving a balance between economic development and environmental sustainability has become one of the major challenges for policymakers and industrial organizations. To ensure sustainable economic growth, it is essential to evaluate organizations from the perspective of their long-term development capabilities. Such evaluations help decision-makers identify enterprises that possess sustainable competitive advantages and strong development potential [4]. Since this evaluation process usually involves multiple criteria and several decision experts, it can naturally be regarded as a multiple-attribute group decision-making (MAGDM) problem. Although numerous MAGDM approaches have been proposed in the literature, existing methods often fail to adequately represent uncertain, vague, and incomplete information provided by experts during the evaluation process. Consequently, there is a growing need for more powerful fuzzy decision-making frameworks capable of handling uncertainty more effectively while providing scientifically reliable support for complex decision-making problems.

Recent advances in multi-criteria decision-making (MCDM) have focused on developing robust fuzzy-based methodologies to address uncertainty in diverse application domains. For instance, Aleksic *et al.,* [5] proposed a DIBR II-fuzzy MABAC model for shooting stance selection, demonstrating the effectiveness of hybrid fuzzy MCDM techniques. Mastilo [6] introduced an entropy-based CORASO framework to evaluate national innovation performance using the European Innovation Scoreboard. Imran *et al.,* [7] developed a fuzzy decision-making framework for assessing the impact of cold email outreach channels, illustrating the applicability of MCDM in business analytics. Similarly, Shahid and Tahir [8] presented an M-polar fuzzy five-way decision-making framework with modified Hamacher aggregation operators for carbon emission reduction assessment. These studies highlight the growing significance of advanced fuzzy MCDM approaches in solving complex real-world decision-making problems under uncertain environments, thereby motivating the development of more flexible and reliable decision-making frameworks.

1.1 Short review of q-rung orthopair fuzzy set

In real-world decision-making problems, experts often encounter situations in which it is difficult to express their preferences using precise numerical values. The presence of vagueness and ambiguity makes classical mathematical models inadequate for representing human judgment. To address this issue, Zadeh introduced the concept of fuzzy sets (FSs), in which uncertainty is described through a membership degree [9]. Since its introduction, fuzzy set theory has become one of the most influential tools for modeling uncertain information in decision-making.

Although FSs effectively describe the degree of membership of an element, they cannot explicitly represent the degree of non-membership or the hesitation of decision-makers. To overcome this limitation, Atanassov introduced intuitionistic fuzzy sets (IFSs), which simultaneously consider membership and non-membership degrees while allowing the hesitation degree to be determined implicitly [10]. The development of IFSs greatly improved the representation of uncertain information and stimulated extensive research in fuzzy decision-making. Over the past decades, intuitionistic fuzzy theory has been successfully applied to numerous practical problems. Nguyen proposed a generalized knowledge-based score function for interval-valued intuitionistic fuzzy sets to improve decision-making performance [11]. Garg and Rani developed several aggregation operators based on complex intuitionistic fuzzy information for multi-criteria decision-making [12]. They further extended these studies by introducing generalized Bonferroni mean aggregation operators under complex intuitionistic fuzzy environments [13]. Deveci *et al.,* presented an interval-valued intuitionistic fuzzy CODAS method for renewable energy selection [14]. Despite these significant developments, IFSs cannot adequately handle situations in which the sum of membership and non-membership degrees exceeds unity. To address this drawback, Yager introduced the concept of Pythagorean fuzzy sets (PFSs), where the sum of the squares of the membership and non-membership degrees is restricted to be less than or equal to one [15]. Compared with IFSs, PFSs provide greater flexibility for representing uncertain information. Following the introduction of PFSs, numerous researchers investigated their applications in decision-making. Xue *et al.,* developed a Pythagorean fuzzy LINMAP method for railway project investment selection [16]. Kumar and Chen further investigated group decision-making methods based on entropy measures and weighted arithmetic mean aggregation operators of Pythagorean fuzzy numbers [17]. Nevertheless, some practical situations still cannot be represented by PFSs. For example, if a decision-maker assigns the membership degree as 0.80 and the non-membership degree as 0.75, then $0.80^2 + 0.75^2 = 1.2025 > 1$, which violates the constraint of PFSs. Hence, such information cannot be modeled within the Pythagorean fuzzy framework. To overcome this limitation, Yager introduced the concept of q-rung orthopair fuzzy sets (q-ROFSs), where the membership and non-membership degrees satisfy the condition $\eta^q + \psi^q \leq 1$ for $q \geq 1$ [18]. The additional parameter q provides significantly greater flexibility than both intuitionistic and Pythagorean fuzzy sets. For instance, when $q = 3$, the values $\eta = 0.80$ and $\psi = 0.75$ satisfy $(0.80)^3 + (0.75)^3 = 0.512 + 0.421875 = 0.933875 < 1$, which demonstrates that the information rejected by PFSs can be successfully represented using the q-ROFS model. Consequently, q-ROFSs offer a more powerful framework for describing uncertainty in complex decision-making problems. Following the introduction of q-ROFSs, several researchers developed new aggregation operators and theoretical models. Xu *et al.,* proposed q-rung dual hesitant fuzzy Heronian mean operators for MAGDM [19]. Wei *et al.,* introduced Heronian mean aggregation operators under the q-ROFS environment [20]. Yager *et al.,* further investigated several fundamental properties of generalized orthopair fuzzy sets [21]. Yang and Pang proposed partitioned Bonferroni mean operators for q-ROFSs [22]. Ye *et al.,* established the theory of differential calculus under the q-ROFS framework [23]. Owing to their superior capability in representing uncertain information, q-ROFSs have been successfully applied in various research fields. Krishankumar *et al.,* employed q-ROFSs in renewable energy source selection [24]. Peng *et al.,* proposed a decision-making framework for mobile edge caching using q-ROFSs [25]. Li *et al.,* applied q-ROFSs to clustering analysis [26]. These studies demonstrate that q-ROFSs provide a more flexible and reliable representation of uncertain information than conventional fuzzy models, making them highly suitable for solving modern multi-attribute decision-making and multi-attribute group decision-making problems.

1.2 The main motivations and contributions

Multi-attribute group decision-making (MAGDM) problems frequently involve complex, vague, and uncertain information that cannot be represented accurately using conventional fuzzy models. Over the past decades, several extensions of fuzzy set theory have been proposed to improve the representation of uncertainty. Intuitionistic fuzzy sets (IFSs) and Pythagorean fuzzy sets (PFSs) have significantly enhanced decision-making models by incorporating both membership and non-membership information. However, these models are defined only on real-valued domains and therefore cannot effectively represent two-dimensional or periodic information that often arises in practical applications. To overcome this limitation, Ramot *et al.,* introduced the concept of complex fuzzy sets (CFSs), where the membership grade is represented by a complex number rather than a real value [27]. This development enabled the representation of two-dimensional uncertain information. Subsequently, Alkouri and Salleh proposed complex intuitionistic fuzzy sets (CIFSs), which simultaneously employ complex membership and non-membership grades within the unit disc [28]. Although CIFSs considerably improved the modeling of complex uncertainty, they still inherited the restrictive intuitionistic constraints and were unable to represent information with higher degrees of uncertainty. Building upon the CIFS framework, Ullah *et al.,* introduced complex Pythagorean fuzzy sets (CPFSs), which relaxed the intuitionistic constraint by adopting the Pythagorean condition [29]. This extension provided greater flexibility for handling uncertain and unreliable information in complex-valued environments. Nevertheless, CPFSs remain limited because the squared sums of the real and imaginary parts of the membership and non-membership grades must satisfy the Pythagorean constraint, which excludes many practical evaluation values encountered in real decision-making problems. For example, suppose that an expert evaluates an alternative G_i under criterion C_j using the complex membership grade $0.95e^{2i\pi(0.83)}$ and the complex non-membership grade $0.88e^{2i\pi(0.86)}$. Such information violates the admissible constraints of both CIFSs and CPFSs and therefore cannot be represented within these frameworks. To address these shortcomings, Liu *et al.,* proposed the theory of complex q-rung orthopair fuzzy sets (Cq-ROFSs) [30]. Unlike IFSs, PFSs, CIFSs, and CPFSs, the Cq-ROFS framework requires only that the qth powers of the real and imaginary parts of the membership and non-membership grades satisfy the orthopair constraint, thereby providing a substantially larger feasible information domain and much greater flexibility for modeling complex uncertain information. Consequently, Cq-ROFSs have become an effective tool for solving challenging MAGDM problems involving complex-valued and highly uncertain data.

Based on the limitations of existing aggregation operators under the Complex q-rung orthopair fuzzy environment and the research gap identified in the literature, this study aims to address the following research questions:

(i) How can new A-complex operational laws be developed under the Complex q-rung orthopair fuzzy (Cq-ROFS) framework to provide a more flexible representation and aggregation of complex-valued uncertain information?

(ii) How can arithmetic and geometric mean aggregation operators be constructed based on the proposed A-complex operational laws while preserving the fundamental characteristics of Cq-ROFS information?

(iii) Do the proposed aggregation operators satisfy desirable mathematical properties such as idempotency, boundedness, and monotonicity, thereby ensuring their theoretical validity and reliability?

(iv) How can the proposed aggregation operators be incorporated into a novel multi-attribute group decision-making (MAGDM) framework for solving complex decision-making problems under the Cq-ROFS environment?

(v) Are the proposed MAGDM approaches effective, robust, and practical for real-world decision-making applications involving complex uncertain information?

(vi) Do the proposed methods provide more flexible information aggregation and more consistent ranking results than existing approaches under the Complex q-rung orthopair fuzzy framework?

1.3 The structure of this paper

According to the following, this article is organized: The main topic of section 2 is the construct of A-complex operational laws based Complex q-ROF weighted arithmetic mean aggregation operator (Cq-ROFWAMAO), Complex q-ROF ordered weighted arithmetic mean aggregation operator (Cq-ROFOWAMAO). The concept of Complex q-ROF weighted geometric mean aggregation operator (Cq-ROFWGMAO) and Complex q-ROF ordered weighted geometric mean aggregation operator (Cq- ROFOWGMAO) within the context of Cq-ROF value sets and their characteristics are covered in Section 3. Section 4 suggests a novel approach of decision-making using the cutting-edge methods based on the Cq-ROFWAMAO and Cq-ROFWGMAO. Additionally, Section 5 provides an example to demonstrate the value of the suggested strategy for making decisions. Section 6 characterizes the comparability and sensitivity analysis that demonstrate the logic and stability of the suggested technique. Section 7 provides a conclusion to the article. Figure 1 presents the graphical representation of the overall section-wise structure of the paper.

2. Complex q-rung orthopair fuzzy weighted arithmetic mean aggregation operators

We provide definitions of A-complex operational laws for Cq-ROFNs in the following section. Following them, several aggregation (Complex q-ROF weighted arithmetic mean aggregation operator (Cq-ROFWAMAO), Complex q-ROF ordered weighted arithmetic mean aggregation operator (Cq-ROFOWAMAO)) operators based on A-complex operational laws will be created.

Definition 1 *([30, 31]) Cq-ROFS is defined by*

$$C = \{(\chi_P(u), \xi_P(u)) : u \in X\}$$

where $\chi_C : X \to \{a_1 + ib_1 : a_1 + ib_1 \in C, |a_1 + ib_1| \leq 1\}$ and $\xi_C : X \to \{a_2 + ib_2 : a_2 + ib_2 \in C, |a_2 + ib_2| \leq 1\}$ such that $\chi_C(u) = a_1 + ib_1$ and $\xi_C(u) = a_2 + ib_2$, provided that $|a_1 + ib_1|^q + |a_2 + ib_2|^q \leq 1$ or $\chi_C(u) = \eta_C(u) e^{i2\pi\left(\eta_C^{im}(u)\right)}$ and $\xi_C(u) = \psi_C(u) e^{i2\pi\left(\psi_C^{im}(u)\right)}$, where $(\eta_C(u))^q + (\psi_C(u))^q \leq 1$ and $(\eta_C^{im}(u))^q + (\psi_C^{im}(u))^q \leq 1$. The Cq-ROFN is defined by $P = \left(\eta_C(u) e^{i2\pi\left(\eta_C^{im}(u)\right)}, \psi_C(u) e^{i2\pi\left(\psi_C^{im}(u)\right)}\right)$.

Definition 2 *([30, 31]) For any Cq-ROFN $P_1 = \left(\eta_{C_1}(u) e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_1}(u) e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)$, the score and accuracy functions are defined by*

$$SC(P_1) = \frac{1}{2}\left\{(\eta_{C_1}(u))^q - (\psi_{C_1}(u))^q + \left(\eta_{C_1}^{im}(u)\right)^q - \left(\psi_{C_1}^{im}(u)\right)^q\right\}$$

and

$$AC(P_1) = \frac{1}{2}\left\{(\eta_{C_1}(u))^q + (\psi_{C_1}(u))^q + \left(\eta_{C_1}^{im}(u)\right)^q + \left(\psi_{C_1}^{im}(u)\right)^q\right\}$$

where $SC(P_1) \in [-1, 1]$ and $AC(P_1) \in [0, 1]$.

Definition 3 *([30, 31]) Let $P_1 = \left(\eta_{C_1}(u) e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_1}(u) e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)$, $P_2 = \left(\eta_{C_2}(u) e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_2}(u) e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)$ be two Cq-ROFNs. Then*

(1) *if $SC(P_1) > SC(P_2)$, then $P_1 > P_2$,*
(2) *if $SC(P_1) = SC(P_2)$ then*
(i) *if $AC(P_1) > AC(P_2)$, then $P_1 > P_2$,*
(ii) *if $AC(P_1) = AC(P_2)$, then $P_1 = P_2$.*

Definition 4 *Let $P_1 = \left(\eta_{C_1}(u) e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_1}(u) e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)$, $P_2 = \left(\eta_{C_2}(u) e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_2}(u) e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)$ be two Cq-ROFNs. Then we define the following algebraic (A)-complex operational laws:*

(i) $P_1 \oplus P_2 = \begin{pmatrix} \left(\frac{1}{2}\left((\eta_{C_1}(u))^q + (\eta_{C_2}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{im}(u)\right)^q + \left(\eta_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(1 - \frac{1}{2}\left((\psi_{C_1}(u))^q + (\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \frac{1}{2}\left(\left(\psi_{C_1}^{im}(u)\right)^q + \left(\psi_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix}$;

(ii) $P_1 \otimes P_2 = \begin{pmatrix} \left(1 - \frac{1}{2}\left((\eta_{C_1}(u))^q + (\eta_{C_2}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \frac{1}{2}\left(\left(\eta_{C_1}^{im}(u)\right)^q + \left(\eta_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(\frac{1}{2}\left((\psi_{C_1}(u))^q + (\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\frac{1}{2}\left(\left(\psi_{C_1}^{im}(u)\right)^q + \left(\psi_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix}$;

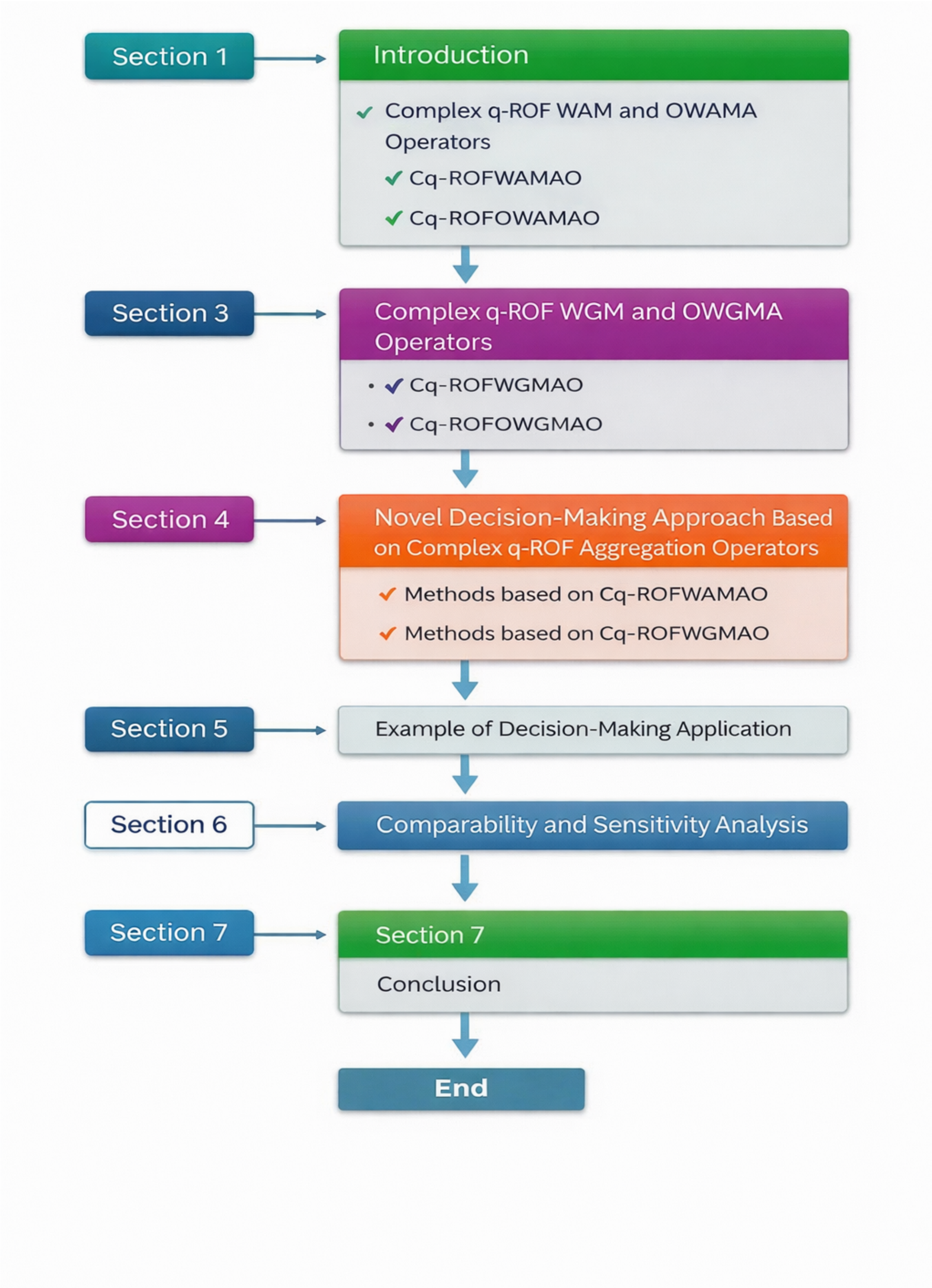

Fig. 1. Graphical representation of the section-wise structure of the paper.

(iii) $\alpha P = \left((\alpha)^{\frac{1}{q}} \eta_C(u) e^{i2\pi(\alpha)^{\frac{1}{q}} \eta_C^{im}(u)}, (\alpha(1-(\eta_C(u))^q))^{\frac{1}{q}} e^{i2\pi\left(\alpha\left(1-\left(\psi_C^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \right)$,
where $0 \leq \alpha \leq 1$;
(iv) $P^\lambda = \left(r^{\frac{1}{q}} (\eta_C(u))^\lambda e^{i2\pi r^{\frac{1}{q}} \left(\eta_C^{im}(u)\right)^\lambda}, r^{\frac{1}{q}} (\eta_C(u))^\lambda e^{i2\pi r^{\frac{1}{q}} \left(\psi_C^{im}(u)\right)^\lambda} \right)$;
(v) $P^{\odot\alpha} = \left((\alpha(1-(\eta_C(u))^q))^{\frac{1}{q}} e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}}, (\alpha)^{\frac{1}{q}} \psi_C(u) e^{i2\pi(\alpha)^{\frac{1}{q}} \psi_C^{im}(u)} \right)$,
where $0 \leq \alpha \leq 1$.

Definition 5 *Let* $\left\{ P_r = \left(\eta_{C_r}(u) e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u) e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)} \right) : r = 1, 2, ..., m \right\}$ *be the collection of* Cq*-ROF values and let* $Cq\text{-}ROFWAM : \Omega^m \to \Omega$. *If* $Cq\text{-}ROFWAM_E(P_1, P_2, P_3, ..., P_m) = \left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus ... \oplus (\alpha_m P_m)^\lambda \right)^{\frac{1}{\lambda}}$ *then* $Cq\text{-}ROFWAM$ *is called a* q*-rung orthopair fuzzy weighted averaging mean operator of dimension* n*, where* Ω *is the set of all* Cq*-ROF values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{ P_r = \left(\eta_{C_r}(u) e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u) e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)} \right) : r = 1, 2, ..., m \right\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$,

$$Cq - ROFWAM_E(P_1, P_2, P_3, ..., P_m)$$
$$= \left(\begin{array}{c} \left(\left(\sum_{r=1}^m \left(\alpha_r (\eta_{C_r}(u))^q\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}}, \\ \left(\left(1 - \sum_{r=1}^m \left(\alpha_r (1-(\psi_{C_r}(u))^q)\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} e^{i2\pi\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}} \end{array} \right)$$

Theorem 1 *Let* $\left\{ P_r = \left(\eta_{C_r}(u) e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u) e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)} \right) : r = 1, 2, ..., m \right\}$ *be the collection of* Cq*-ROF values. Then by using the* $Cq\text{-}ROFWAM_E$ *operator their aggregated value is also a* Cq*-ROF value and*

$$Cq\text{-}ROFWAM_E(P_1, P_2, P_3, ..., P_m)$$
$$= \left(\begin{array}{c} \left(\left(\sum_{r=1}^m \left(\alpha_r (\eta_{C_r}(u))^q\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^m \left(\alpha_r \left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}}, \\ \left(\left(1 - \sum_{r=1}^m \left(\alpha_r (1-(\psi_{C_r}(u))^q)\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} e^{i2\pi\left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}}} \end{array} \right)$$

$E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* $\left\{ P_r = \left(\eta_{C_r}(u) e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u) e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)} \right) : r = 1, 2, ..., n \right\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^n \alpha_r = 1$.

Proof 1 *Let*

$$\alpha_1 P_1 = \left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}(u) e^{i2\pi(\alpha_1)^{\frac{1}{q}} \eta_{C_1}^{im}(u)}, (\alpha_1(1-(\psi_{C_1}(u))^q))^{\frac{1}{q}} e^{i2\pi\left(\alpha_1\left(1-\left(\psi_{C_1}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \right)$$

and

$$\alpha_2 P_2=\left((\alpha_2)^{\frac{1}{q}}\eta_{C_2}(u)\,e^{i2\pi(\alpha_2)^{\frac{1}{q}}\eta^{im}_{C_2}(u)},\left(\alpha_2\left(1-(\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}}e^{i2\pi\left(\alpha_2\left(1-\left(\psi^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}}\right).$$

Then

$$(\alpha_1P_1)^\lambda=\begin{pmatrix} r^{\frac{1}{q}}\left((\alpha_1)^{\frac{1}{q}}\eta_{C_1}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left((\alpha_1)^{\frac{1}{q}}\eta^{im}_{C_1}(u)\right)^\lambda}, \\ r^{\frac{1}{q}}\left(\left(\alpha_1\left(1-(\psi_{C_1}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\psi^{im}_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda} \end{pmatrix}$$

and

$$(\alpha_2P_2)^\lambda=\begin{pmatrix} r^{\frac{1}{q}}\left((\alpha_2)^{\frac{1}{q}}\eta_{C_2}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left((\alpha_2)^{\frac{1}{q}}\eta^{im}_{C_2}(u)\right)^\lambda}, \\ r^{\frac{1}{q}}\left(\left(\alpha_2\left(1-(\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_2\left(1-\left(\psi^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda} \end{pmatrix}.$$

Now

$$\begin{aligned}&(\alpha_1P_1)^\lambda\oplus(\alpha_2P_2)^\lambda\\ =&\begin{pmatrix} \left(\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left((\alpha_1)^{\frac{1}{q}}\eta_{C_1}(u)\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left((\alpha_2)^{\frac{1}{q}}\eta_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left((\alpha_1)^{\frac{1}{q}}\eta^{im}_{C_1}(u)\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left((\alpha_2)^{\frac{1}{q}}\eta^{im}_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(1-\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\left(1-(\psi_{C_1}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\left(1-(\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\psi^{im}_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\left(1-\left(\psi^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix}\end{aligned}$$

$$=\begin{pmatrix} \left(\frac{1}{2}\left(2\left(\left((\alpha_1)^{\frac{1}{q}}\eta_{C_1}(u)\right)^\lambda\right)^q+2\left(\left((\alpha_2)^{\frac{1}{q}}\eta_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(\frac{1}{2}\left(2\left(\left((\alpha_1)^{\frac{1}{q}}\eta^{im}_{C_1}(u)\right)^\lambda\right)^q+2\left(\left((\alpha_2)^{\frac{1}{q}}\eta^{im}_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \\ ,\left(1-\frac{1}{2}\left(2\left(\left(\left(\alpha_1\left(1-(\psi_{C_1}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\left(1-(\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\frac{1}{2}\left(2\left(\left(\left(\alpha_1\left(1-\left(\psi^{im}_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\left(1-\left(\psi^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

$$=\begin{pmatrix} \left(\left(\left((\alpha_1)^{\frac{1}{q}}\eta_{C_1}(u)\right)^\lambda\right)^q+\left(\left((\alpha_2)^{\frac{1}{q}}\eta_{C_2}(u)\right)^\lambda\right)^q\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\left((\alpha_1)^{\frac{1}{q}}\eta^{im}_{C_1}(u)\right)^\lambda\right)^q+\left(\left((\alpha_2)^{\frac{1}{q}}\eta^{im}_{C_2}(u)\right)^\lambda\right)^q\right)^{\frac{1}{q}}} \\ ,\left(1-\left(\left(\left(\left(\alpha_1\left(1-(\psi_{C_1}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(\left(\left(\alpha_2\left(1-(\psi_{C_2}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\left(\left(\left(\left(\alpha_1\left(1-\left(\psi^{im}_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(\left(\left(\alpha_2\left(1-\left(\psi^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

$$= \left(\begin{array}{c} \left(\left(\left((\alpha_1)^{\frac{1}{q}} \eta_{C_1}(u) \right)^q \right)^\lambda + \left(\left((\alpha_2)^{\frac{1}{q}} \eta_{C_2}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(\left(\left((\alpha_1)^{\frac{1}{q}} \eta^{im}_{C_1}(u) \right)^q \right)^\lambda + \left(\left((\alpha_2)^{\frac{1}{q}} \eta^{im}_{C_2}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}}} \\ , \left(1 - \left(\left(\left(\left(\alpha_1 \left(1 - (\psi_{C_1}(u))^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left(\left(\alpha_2 \left(1 - (\psi_{C_2}(u))^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}} \\ e^{i2\pi \left(1 - \left(\left(\left(\left(\alpha_1 \left(1 - \left(\psi^{im}_{C_1}(u) \right)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q + \left(\left(\left(\alpha_2 \left(1 - \left(\psi^{im}_{C_2}(u) \right)^q \right) \right)^{\frac{1}{q}} \right)^\lambda \right)^q \right) \right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left((\alpha_1 (\eta_{C_1}(u))^q)^\lambda + (\alpha_2 (\eta_{C_2}(u))^q)^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(\left(\alpha_1 \left(\eta^{im}_{C_1}(u) \right)^q \right)^\lambda + \left(\alpha_2 \left(\eta^{im}_{C_2}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}}} \\ , \left(1 - \left((\alpha_1 (1 - (\psi_{C_1}(u))^q))^\lambda + (\alpha_2 (1 - (\psi_{C_2}(u))^q))^\lambda \right) \right)^{\frac{1}{q}} \\ e^{i2\pi \left(1 - \left(\left(\alpha_1 \left(1 - \left(\psi^{im}_{C_1}(u) \right)^q \right) \right)^\lambda + \left(\alpha_2 \left(1 - \left(\psi^{im}_{C_2}(u) \right)^q \right) \right)^\lambda \right) \right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\sum_{r=1}^{2} (\alpha_r (\eta_{C_r}(u))^q)^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(\sum_{r=1}^{2} \left(\alpha_r \left(\eta^{im}_{C_r}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}}}, \\ \left(1 - \sum_{r=1}^{2} (\alpha_r (1 - (\psi_{C_r}(u))^q))^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(1 - \sum_{r=1}^{2} \left(\alpha_r \left(1 - \left(\psi^{im}_{C_r}(u) \right)^q \right) \right)^\lambda \right)^{\frac{1}{q}}} \end{array} \right).$$

Thus

$$\left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \right) = \left(\begin{array}{c} \left(\sum_{r=1}^{2} (\alpha_r (\eta_{C_r}(u))^q)^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(\sum_{r=1}^{2} \left(\alpha_r \left(\eta^{im}_{C_r}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}}}, \\ \left(1 - \sum_{r=1}^{2} (\alpha_r (1 - (\psi_{C_r}(u))^q))^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(1 - \sum_{r=1}^{2} \left(\alpha_r \left(1 - \left(\psi^{im}_{C_r}(u) \right)^q \right) \right)^\lambda \right)^{\frac{1}{q}}} \end{array} \right).$$

Let suppose that the result is true for $r = k$.

$$\left((\alpha_1 P_1)^\lambda \oplus (\alpha_2 P_2)^\lambda \oplus (\alpha_3 P_3)^\lambda \oplus \ldots \oplus (\alpha_k P_k)^\lambda \right)$$

$$= \left(\begin{array}{c} \left(\sum_{r=1}^{k} (\alpha_r (\eta_{C_r}(u))^q)^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(\sum_{r=1}^{k} \left(\alpha_r \left(\eta^{im}_{C_r}(u) \right)^q \right)^\lambda \right)^{\frac{1}{q}}}, \\ \left(1 - \sum_{r=1}^{k} (\alpha_r (1 - (\psi_{C_r}(u))^q))^\lambda \right)^{\frac{1}{q}} e^{i2\pi \left(1 - \sum_{r=1}^{k} \left(\alpha_r \left(1 - \left(\psi^{im}_{C_r}(u) \right)^q \right) \right)^\lambda \right)^{\frac{1}{q}}} \end{array} \right).$$

We shows that the result is true for $r=k+1$. *So*

$$\left((\alpha_1P_1)^\lambda\oplus(\alpha_2P_2)^\lambda\oplus(\alpha_3P_3)^\lambda\oplus...\oplus(\alpha_kP_k)^\lambda\right)\oplus(\alpha_{k+1}P_{k+1})^\lambda$$

$$=\left(\begin{array}{c}\left(\sum_{r=1}^{k}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}e^{i2\pi\left(\sum_{r=1}^{k}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}},\\ \left(1-\sum_{r=1}^{k}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}e^{i2\pi\left(1-\sum_{r=1}^{k}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}r^{\frac{1}{q}}\left((\alpha_{k+1})^{\frac{1}{q}}\eta_{C_{k+1}}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left((\alpha_{k+1})^{\frac{1}{q}}\eta_{C_{k+1}}^{im}(u)\right)^\lambda},\\ r^{\frac{1}{q}}\left(\left(\alpha_{k+1}\left(1-\left(\psi_{C_{k+1}}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_{k+1}\left(1-\left(\psi_{C_{k+1}}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}e^{i2\pi\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}},\\ \left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}e^{i2\pi\left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}\end{array}\right).$$

Thus

$$\left(\left((\alpha_1P_1)^\lambda\oplus(\alpha_2P_2)^\lambda\oplus(\alpha_3P_3)^\lambda\oplus...\oplus(\alpha_kP_k)^\lambda\right)\oplus(\alpha_{k+1}P_{k+1})^\lambda\right)^{\frac{1}{\lambda}}$$

$$=.\left(\begin{array}{c}\left(\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}},\\ \left(\left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}e^{i2\pi\left(\left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}\end{array}\right).$$

Theorem 2 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq-*ROF values. Let* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$. *If* $\left(\eta_{C_1}(u)\,e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)},\psi_{C_1}(u)\,e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)=\left(\eta_{C_2}(u)\,e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}\right)=...$ $=\left(\eta_{C_m}(u)\,e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)},\psi_{C_m}(u)\,e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta_{C}^{im}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi_{C}^{im}(u)\right)}\right)$ *and* $\lambda=1$, *then* $Cq\text{-}ROFWAM_E(P_1,P_2,P_3,...,P_m)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta_{C}^{im}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi_{C}^{im}(u)\right)}\right)$.

Proof 2 *Let* $\left(\eta_{C_1}(u)\,e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)},\psi_{C_1}(u)\,e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)=\left(\eta_{C_2}(u)\,e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}\right)=...$ $=\left(\eta_{C_m}(u)\,e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)},\psi_{C_m}(u)\,e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta_{C}^{im}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi_{C}^{im}(u)\right)}\right)$ *and* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\{P_r=(\mu_r,\upsilon_r):r=1,2,...,m\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=$

1. ***Based on Definition 5, we get***

$$
\begin{aligned}
& Cq-ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \\
= & \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \end{pmatrix} \\
= & \begin{pmatrix} \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_C(u)\right)^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_C^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_C(u)\right)^q\right)\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_C^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \end{pmatrix} \\
= & \begin{pmatrix} \left(\left(\left(\eta_C(u)\right)^q\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\left(\eta_C^{im}(u)\right)^q\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)^{\frac{1}{q}}}, \\ \left(1-\left(1-\left(\psi_C(u)\right)^q\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\psi_C^{im}(u)\right)^q\right)\sum_{r=1}^{m}\left(\alpha_r\right)\right)^{\frac{1}{q}}} \end{pmatrix} \\
= & \left(\left(\left(\left(\eta_C(u)\right)^q\right)^{\frac{1}{q}}\right) e^{i2\pi\left(\left(\eta_C^{im}(u)\right)^q\right)^{\frac{1}{q}}}, \left(1-\left(1-\left(\psi_C(u)\right)^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\psi_C^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}}\right) \\
= & \left(\eta_C(u)\, e^{i2\pi\eta_C^{im}(u)}, \psi_C(u)\, e^{i2\pi\psi_C^{im}(u)}\right).
\end{aligned}
$$

Theorem 3 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right) : r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values and* $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right) : r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$. *Let* $P^-=\left(\eta_C^-(u)\, e^{i2\pi\left(\eta_C^{im-}(u)\right)}, \psi_C^-(u)\, e^{i2\pi\left(\psi_C^{im-}(u)\right)}\right)$ *and let* $P^+=\left(\eta_C^+(u)\, e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right)}, \psi_{C_r}^+(u)\, e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)}\right)$, *where*

$\eta_C^-(u)\, e^{i2\pi\left(\eta_C^{im-}(u)\right),}=$ min $\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta_{C_1}^{im}(u)\right),}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta_{C_2}^{im}(u)\right),}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta_{C_m}^{im}(u)\right),}\right\}$,

$\psi_C^-(u)\, e^{i2\pi\left(\psi_C^{im-}(u)\right)}=$ max $\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$,

$\eta_C^+(u)\, e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right),}=$ max $\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta_{C_1}^{im}(u)\right),}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta_{C_2}^{im}(u)\right),}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta_{C_m}^{im}(u)\right),}\right\}$ *and*

$\psi_{C_r}^+(u)\, e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)}=$ min $\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$ *and* $\lambda=1$. *Then* $P^-\leq Cq-ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right)\leq P^+$.

Proof 3 *As* $P^-=\left(\eta_C^-(u)\, e^{i2\pi\left(\eta_C^{im-}(u)\right)}, \psi_C^-(u)\, e^{i2\pi\left(\psi_C^{im-}(u)\right)}\right)$ *and let* $P^+=\left(\eta_C^+(u)\, e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right)}, \psi_{C_r}^+(u)\, e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)}\right)$, *where*

$\eta_C^-(u)\, e^{i2\pi\left(\eta_C^{im-}(u)\right),}=$ min $\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta_{C_1}^{im}(u)\right),}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta_{C_2}^{im}(u)\right),}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta_{C_m}^{im}(u)\right),}\right\}$,

$\psi_C^-(u)\, e^{i2\pi\left(\psi_C^{im-}(u)\right)}=$ max $\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$,

$\eta_C^+(u)\, e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right),}=$ max $\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta_{C_1}^{im}(u)\right),}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta_{C_2}^{im}(u)\right),}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta_{C_m}^{im}(u)\right),}\right\}$ *and*

$\psi_{C_r}^{+}(u)\, e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)} = \min\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$, *we get* $(\eta_{C_t}(u))^q \leq \left(\eta_C^{+}(u)\right)^q$ *where,*$t = 1, 2, ..., n$. *Then we have* $\alpha_r(\eta_{C_t}(u))^q \leq \alpha_r\left(\eta_C^{+}(u)\right)^q$. *This implies that* $\sum_{r=1}^{m} \alpha_r(\eta_{C_r}(u))^q \leq \sum_{r=1}^{m} \alpha_r\left(\eta_C^{+}(u)\right)^q$ *then we have,* $\sum_{r=1}^{m} \alpha_r(\eta_{C_r}(u))^q \leq \left(\eta_C^{+}(u)\right)^q \sum_{r=1}^{m} \alpha_r$. *This implies that* $\sum_{r=1}^{m} \alpha_r(\eta_{C_r}(u))^q \leq \left(\eta_C^{+}(u)\right)^q$ *and this implies that* $\left(\sum_{r=1}^{m} \alpha_r(\eta_{C_r}(u))^q\right)^{\frac{1}{q}} \leq \eta_C^{+}(u)$.
Also

$$\left(\sum_{r=1}^{m} \alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^{\frac{1}{q}} \leq \eta_C^{im+}(u) \Rightarrow e^{i2\pi\left(\sum_{r=1}^{m} \alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^{\frac{1}{q}}} \leq e^{i2\pi\eta_C^{im+}(u)}.$$

Thus

$$\left(\sum_{r=1}^{m} \alpha_r(\eta_{C_r}(u))^q\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{m} \alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^{\frac{1}{q}}} \leq \eta_C^{+}(u)\, e^{i2\pi\eta_C^{im+}(u)}.$$

On the other hand, we get $(\psi_{C_r}(u))^q \geq \left(\psi_C^{+}(u)\right)^q$. *This implies that* $-(\psi_{C_r}(u))^q \leq -\left(\psi_C^{+}(u)\right)^q$, *also* $1-(\psi_{C_r}(u))^q \leq 1-\left(\psi_C^{+}(u)\right)^q$, *where,* $r = 1, 2, ..., n$. *Then we have* $\alpha_r(1-(\psi_{C_r}(u))^q) \leq \alpha_r\left(1-\left(\psi_C^{+}(u)\right)^q\right)$ *then we get,*

$$\begin{aligned}
\alpha_r(1-(\psi_{C_r}(u))^q) &\leq \alpha_r\left(1-\left(\psi_C^{+}(u)\right)^q\right) \\
&\Rightarrow \sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q) \leq \sum_{r=1}^{m} \alpha_r\left(1-\left(\psi_C^{+}(u)\right)^q\right) \\
&\Rightarrow \sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q) \leq \left(1-\left(\psi_C^{+}(u)\right)^q\right)\sum_{r=1}^{m} \alpha_r \\
&\Rightarrow -\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q) \geq -\left(1-\left(\psi_C^{+}(u)\right)^q\right)\sum_{r=1}^{m} \alpha_r \\
&\Rightarrow 1-\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q) \geq 1-\left(1-\left(\psi_C^{+}(u)\right)^q\right)\sum_{r=1}^{m} \alpha_r.
\end{aligned}$$

Also

$$\begin{aligned}
\left(1-\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q)\right)^{\frac{1}{q}} &\geq \left(1-\left(1-\left(\psi_C^{+}(u)\right)^q\right)\sum_{r=1}^{m} \alpha_r\right)^{\frac{1}{q}} \\
&\Rightarrow \left(1-\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q)\right)^{\frac{1}{q}} \geq \left(1-\left(1-\left(\psi_C^{+}(u)\right)^q\right)\right)^{\frac{1}{q}} \\
&\Rightarrow \left(1-\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q)\right)^{\frac{1}{q}} \geq \left(\left(\psi_C^{+}(u)\right)^q\right)^{\frac{1}{q}} \\
&\Rightarrow \left(1-\sum_{r=1}^{m} \alpha_r(1-(\psi_{C_r}(u))^q)\right)^{\frac{1}{q}} \geq \psi_C^{+}(u).
\end{aligned}$$

Further

$$\begin{aligned}\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}} &\geq \left(1-\left(1-\left(\upsilon^{im+}\right)^q\right)\sum_{r=1}^{m}\alpha_r\right)^{\frac{1}{q}}\\ &\Rightarrow \left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}} \geq \left(1-\left(1-\left(\upsilon^{im+}\right)^q\right)\right)^{\frac{1}{q}}\\ &\Rightarrow \left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}} \geq \left(\left(\upsilon^{im+}\right)^q\right)^{\frac{1}{q}}\\ &\Rightarrow \left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}} \geq \upsilon^{im+}\\ &\Rightarrow e^{i2\pi\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \geq e^{i2\pi\upsilon^{im+}}.\end{aligned}$$

Thus

$$\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}} \geq \psi_C^{+}(u)\, e^{i2\pi\psi_C^{im+}(u)}.$$

Based on Definition 3 and Definition 5, we get $Cq\text{-}ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \leq P^{+}$*. Similarly, we can have* $Cq\text{-}ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \geq P^{-}$*. Finally we get,*

$$P^{+} \geq Cq-ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \geq P^{-}.$$

Theorem 4 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *and* $\left\{P_r^{*}=\left(\eta_{C_r}^{*}(u)\,e^{i2\pi\left(\eta_{C_r}^{im*}(u)\right)},\psi_{C_r}^{*}(u)\,e^{i2\pi\left(\psi_{C_r}^{im*}(u)\right)}\right):r=1,2,...,m\right\}$ *are two collections of* Cq*-ROF values. If* $\eta_{C_r}(u) \leq \eta_{C_r}^{*}(u)$, $\eta_{C_r}^{im}(u) \leq \eta_{C_r}^{im*}(u)$, $\psi_{C_r}(u) \geq \psi_{C_r}^{*}(u)$ *and* $\psi_{C_r}^{im}(u) \geq \psi_{C_r}^{im*}(u)$ *where* $r=1,2,...,m$*, then* $Cq\text{-}ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right)$ $\leq Cq-ROFWAM_E\left(P_1^{*},P_2^{*},P_3^{*},...,P_m^{*}\right)$.

Proof 4 *Given that* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *and* $\left\{P_r^{*}=\left(\eta_{C_r}^{*}(u)\,e^{i2\pi\left(\eta_{C_r}^{im*}(u)\right)},\psi_{C_r}^{*}(u)\,e^{i2\pi\left(\psi_{C_r}^{im*}(u)\right)}\right):r=1,2,...,m\right\}$ *are two collections of* Cq*-ROF values. If* $\eta_{C_r}(u) \leq \eta_{C_r}^{*}(u)$, , $\psi_{C_r}(u) \geq \psi_{C_r}^{*}(u)$ *and* $\psi_{C_r}^{im}(u) \geq \psi_{C_r}^{im+}(u)$ *where* $r=1,2,...,m$*, then* $\left(\eta_{C_r}(u)\right)^q \leq \left(\eta_{C_r}^{*}(u)\right)^q$ *and* $\left(\psi_{C_r}(u)\right)^q \geq \left(\psi_{C_r}^{*}(u)\right)^q$ *. As we have*

$$\begin{aligned}\left(\eta_{C_r}(u)\right)^q &\leq \left(\eta_{C_r}^{*}(u)\right)^q\\ &\Rightarrow \alpha_r\left(\eta_{C_r}(u)\right)^q \leq \alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\\ &\Rightarrow \left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^{\lambda}\\ &\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^{\lambda}\end{aligned}$$

$$\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^*_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^*_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \qquad \text{(i)}$$

Also

$$\begin{aligned}
\left(\eta^{im}_{C_r}(u)\right)^q &\leq \left(\eta^{im*}_{C_r}(u)\right)^q \leq \\
&\Rightarrow \alpha_r\left(\eta^{im}_{C_r}(u)\right)^q \leq \alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q \\
&\Rightarrow \left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda} \leq \left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda} \leq \left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda}
\end{aligned}$$

$$\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}$$

$$\Rightarrow e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \qquad \text{(ii)}$$

From (i) *and* (ii)

$$\begin{aligned}
&\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \\
\leq &\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^*_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{im*}_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \qquad (A)
\end{aligned}$$

Now

$$\begin{aligned}
\left(\psi_{C_r}(u)\right)^q &\geq \left(\psi^*_{C_r}(u)\right)^q \\
&\Rightarrow -\left(\psi_{C_r}(u)\right)^q \leq -\left(\psi^*_{C_r}(u)\right)^q \\
&\Rightarrow \left(1-\left(\psi_{C_r}(u)\right)^q\right) \leq \left(1-\left(\psi^*_{C_r}(u)\right)^q\right) \\
&\Rightarrow \alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right) \leq \alpha_r\left(1-\left(\psi^*_{C_r}(u)\right)^q\right) \\
&\Rightarrow \left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^{\lambda} \leq \left(\alpha_r\left(1-\left(\psi^*_{C_r}(u)\right)^q\right)\right)^{\lambda}.
\end{aligned}$$

Then we have

$$\begin{aligned} & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda \\ \leq & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}} \\ \Rightarrow & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \quad (\text{iii}). \end{aligned}$$

Also

$$\begin{aligned} & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda \\ \leq & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda \\ \Rightarrow & \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}} \\ \Rightarrow & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} \\ \Rightarrow & e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \geq e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \quad (\text{iv}) \end{aligned}$$

From (iii) *and* (iv) , *we get*

$$\begin{aligned} & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \\ \geq & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^*(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \quad (B). \end{aligned}$$

Thus, from (A) *and* (B) *, we have*

$$\left(\begin{array}{c}\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}\end{array}\right)$$

$$\leq\left(\begin{array}{c}\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}\end{array}\right).$$

Let $Cq\text{-}ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right) = T$ *and* $Cq\text{-}ROFWAM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right) = T^*$. *Then, we have* $S(T)\leq S(T^*)$ *. Now there are two possiblity which are discussed one by one.*

(1) *If* $S(T)<S(T^*)$ *, then by Definition 3*, *we have*

$$Cq-ROFWAM_E\left(P_1,P_2,P_3,...,P_m\right)<Cq-ROFWAM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right).$$

(2) *If* $S(T)=S(T^*)$ *, then we have*

$$\left(\begin{array}{c}\left(\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q} e^{i2\pi\left(\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}} - \\ \left(\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q} e^{i2\pi\left(\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q} e^{i2\pi\left(\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}} - \\ \left(\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q} e^{i2\pi\left(\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}}\end{array}\right).$$

Because $\eta_{C_r}(u)\leq\eta_{C_r}^{*}(u)\,\mu_r^*,\eta_{C_r}^{im}(u)\leq\eta_{C_r}^{im*}(u),\psi_{C_r}(u)\geq\psi_{C_r}^{*}(u)$ *and* $\psi_{C_r}^{im}(u)\geq\psi_{C_r}^{im*}(u)$ *, then*

$$\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}=\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}$$

and

$$\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}=\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^{q}.$$

Similarly the imaginary parts are

$$\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q=\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q$$

and

$$\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q=\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q.$$

Now

$$\begin{aligned}H(T) &= \frac{1}{2}\left(\begin{array}{c}\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q+\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q\\+\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q+\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q\end{array}\right)\\&= \frac{1}{2}\left(\begin{array}{c}\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q+\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q\\+\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q+\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}\right)^q\end{array}\right)\\&= H(T^*).\end{aligned}$$

Definition 6 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values and let* Cq*-*$ROFOWAM:\Omega^m\rightarrow\Omega$*, if* Cq*-*$ROFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\left(\alpha_1P_{\delta(1)}\right)^{\lambda}\oplus\left(\alpha_2P_{\delta(1)}\right)^{\lambda}\oplus\left(\alpha_3P_{\delta(3)}\right)^{\lambda}\oplus...\oplus\left(\alpha_mP_{\delta(m)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$ *then* Cq*-*$ROFOWAM$ *is called a* Cq*-rung orthopair fuzzy ordered weighted averaging mean operator of dimension* n*, where* $\left(\delta(1),\delta(2),...,\delta(m)\right)$ *is a permutation of* $(1,2,...,m)$ *such that* $P_{\delta(r-1)}\geq P_{\delta(r)}$ *for all* r*,* Ω *is the set of all* Cq*-ROF values,* $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1.$

Theorem 5 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values. Then by using the* Cq*-*$ROFOWAM_E$ *operator their aggregated value is also a* Cq*-ROF value and*

$$\begin{aligned}&Cq-ROFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)\\&=\left(\begin{array}{c}\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}^{im}(u)\right)^q\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}},\\\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_{\delta(r)}}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_{\delta(r)}}^{im}(u)\right)^q\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}\end{array}\right)\end{aligned}$$

$E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{n}\alpha_r=1$.

Proof 5 *Similar to the proof of Theorem 1.*

Theorem 6 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values. Let* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$. *If* $\left(\eta_{C_1}(u)\,e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)},\psi_{C_1}(u)\,e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right)=\left(\eta_{C_2}(u)\,e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}\right)=...$ $=\left(\eta_{C_m}(u)\,e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)},\psi_{C_m}(u)\,e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta_{C}^{im}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi_{C}^{im}(u)\right)}\right)$ *and* $\lambda=1$ *then* $Cq\text{-}ROFOWAM_E(P_1,P_2,P_3,...,P_m)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta_{C}^{im}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi_{C}^{im}(u)\right)}\right)$.

Proof 6 *Similar to the proof of Theorem 2.*

Theorem 7 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* q*-ROF values and* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$. *Let* $P^-=\left(\eta_C^-(u)\,e^{i2\pi\left(\eta_C^{im-}(u)\right)},\psi_C^-(u)\,e^{i2\pi\left(\psi_C^{im-}(u)\right)}\right)$ *and let* $P^+=\left(\eta_C^+(u)\,e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right)},\psi_{C_r}^+(u)\,e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)}\right)$*, where* $\eta_C^-(u)\,e^{i2\pi\left(\eta_C^{im-}(u)\right)},=$ min $\left\{\eta_{C_1}(u)\,e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)},\eta_{C_2}(u)\,e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)},...,\eta_{C_m}(u)\,e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)},\right\}$,
$\psi_C^-(u)\,e^{i2\pi\left(\psi_C^{im-}(u)\right)}=$ max $\left\{\psi_{C_1}(u)\,e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)},...,\psi_{C_m}(u)\,e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$,
$\eta_C^+(u)\,e^{i2\pi\left(\eta_{C_r}^{im+}(u)\right)},=$ max $\left\{\eta_{C_1}(u)\,e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)},\eta_{C_2}(u)\,e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)},...,\eta_{C_m}(u)\,e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)},\right\}$ *and*
$\psi_{C_r}^+(u)\,e^{i2\pi\left(\psi_{C_r}^{im+}(u)\right)}=$ min $\left\{\psi_{C_1}(u)\,e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)},...,\psi_{C_m}(u)\,e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right\}$ *and* $\lambda=1$. *Then* $P^-\leq Cq-ROFOWAM_E(P_1,P_2,P_3,...,P_m)\leq P^+$.

Proof 7 *Similar to the proof of Theorem 3.*

Theorem 8 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *and* $\left\{P_r^*=\left(\eta_{C_r}^*(u)\,e^{i2\pi\left(\eta_{C_r}^{im*}(u)\right)},\psi_{C_r}^*(u)\,e^{i2\pi\left(\psi_{C_r}^{im*}(u)\right)}\right):r=1,2,...,m\right\}$ *are two collections of* Cq*-ROF values. If* $\eta_{C_r}(u)\leq\eta_{C_r}^*(u)$, $\eta_{C_r}^{im}(u)\leq\eta_{C_r}^{im*}(u)$, $\psi_{C_r}(u)\geq\psi_{C_r}^*(u)$ *and* $\psi_{C_r}^{im}(u)\geq\psi_{C_r}^{im*}(u)$ *where* $r=1,2,...,m$, *then* $Cq\text{-}ROFOWAM_E(P_1,P_2,P_3,...,P_m)$ $\leq Cq\text{-}ROFOWAM_E(P_1^*,P_2^*,P_3^*,...,P_m^*)$.

Proof 8 *Similar to the proof of Theorem 4.*

3. Complex q-rung orthopair fuzzy weighted geometric mean aggregation operators

Here, we provide some new geometric aggregation operators, Complex q-ROF weighted geometric mean aggregation operator (Cq-ROFWGMAO) and Complex q-ROF ordered weighted geometric mean aggregation operator (Cq-ROFOWGMAO), using the suggested operations.

Definition 7 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* q*-ROF values and let* $Cq\text{-}ROFWGM : \Omega^m \to \Omega$*, if* $Cq\text{-}ROFWGM_E(P_1, P_2, P_3, ..., P_m) = \left(\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda \otimes \left(P_3^{\odot\alpha_3}\right)^\lambda \otimes ... \otimes \left(P_m^{\odot\alpha_m}\right)^\lambda\right)^{\frac{1}{\lambda}}$ *then* $Cq\text{-}ROFWGM$ *is called a comlex* q*-rung orthopair fuzzy weighted geometric mean operator of dimension* n*, where* Ω *is the set of all* q*-ROF values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$.

Theorem 9 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* Cq*-ROF values. Then by using the* $Cq\text{-}ROFWGM_E$ *operator their aggregated value is also a* Cq*-ROF value and*

$$Cq - ROFWGM_E(P_1, P_2, P_3, ..., P_m)$$
$$= \left(\begin{array}{c} \left(\left(1 - \sum_{r=1}^m \left(\alpha_r \left(1 - (\eta_{C_r}(u))^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1 - \sum_{r=1}^m \left(\alpha_r\left(1-\left(\eta^{im}_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(\sum_{r=1}^m \left(\alpha_r (\psi_{C_r}(u))^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^m \left(\alpha_r\left(\psi^{im}_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \end{array}\right)$$

$E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* $\left\{P_r \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^n \alpha_r = 1$.

Proof 9 *Let*

$$P_1^{\odot\alpha_1} = \left(\left(\alpha\left(1 - (\eta_{C_1}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\alpha\left(1-\left(\eta^{im}_{C_1}(u)\right)^q\right)\right)}, (\alpha)^{\frac{1}{q}}\, \psi_{C_1}(u)\, e^{i2\pi(\alpha)^{\frac{1}{q}}\psi^{im}_{C_1}(u)}\right)$$

and

$$P_2^{\odot\alpha_2} = \left(\left(\alpha\left(1 - (\eta_{C_2}(u))^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\alpha\left(1-\left(\eta^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}}, (\alpha)^{\frac{1}{q}}\, \psi_{C_2}(u)\, e^{i2\pi(\alpha)^{\frac{1}{q}}\psi^{im}_{C_2}(u)}\right)$$

Then

$$\left(P_1^{\odot\alpha_1}\right)^\lambda = \left(\begin{array}{c} r^{\frac{1}{q}}\left(\left(\alpha_1\left(1 - (\eta_{C_1}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\eta^{im}_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda}, \\ r^{\frac{1}{q}}\left((\alpha)^{\frac{1}{q}}\, \psi_{C_1}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left((\alpha)^{\frac{1}{q}}\psi^{im}_{C_1}(u)\right)^\lambda} \end{array}\right)$$

and

$$\left(P_2^{\odot\alpha_2}\right)^\lambda = \left(\begin{array}{c} r^{\frac{1}{q}}\left(\left(\alpha_1\left(1 - (\eta_{C_2}(u))^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\eta^{im}_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda}, \\ r^{\frac{1}{q}}\left((\alpha)^{\frac{1}{q}}\, \psi_{C_2}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left((\alpha)^{\frac{1}{q}}\psi^{im}_{C_2}(u)\right)^\lambda} \end{array}\right).$$

Now

$$\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda$$

$$= \left(\begin{array}{c} \left(1-\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\eta_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\left(1-\left(\eta_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\left(1-\left(\eta_{C_1}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\left(1-\left(\eta_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}}, \\ ,\left(\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}(u)\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(\frac{1}{2}\left(\left(2^{\frac{1}{q}}\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}^{im}(u)\right)^\lambda\right)^q+\left(2^{\frac{1}{q}}\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}^{im}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1-\frac{1}{2}\left(2\left(\left(\left(\alpha_1\left(1-\left(\eta_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\left(1-\left(\eta_{C_2}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\frac{1}{2}\left(2\left(\left(\left(\alpha_1\left(1-\left(\eta_{C_1}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\left(1-\left(\eta_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(\frac{1}{2}\left(2\left(\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}(u)\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(\frac{1}{2}\left(2\left(\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}^{im}(u)\right)^\lambda\right)^q+2\left(\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}^{im}(u)\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1-\left(\left(\left(\left(\alpha_1\left(1-\left(\eta_{C_1}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(\left(\left(\alpha_2\left(1-\left(\mu_2\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\left(\left(\left(\left(\alpha_1\left(1-\left(\eta_{C_1}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q+\left(\left(\left(\alpha_2\left(1-\left(\eta_{C_2}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda\right)^q\right)\right)^{\frac{1}{q}}}, \\ \left(\left(\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}(u)\right)^q\right)^\lambda+\left(\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\left(\left(\alpha_1\right)^{\frac{1}{q}}\psi_{C_1}^{im}(u)\right)^q\right)^\lambda+\left(\left(\left(\alpha_2\right)^{\frac{1}{q}}\psi_{C_2}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1-\sum_{r=1}^{2}\left(\alpha_r\left(1-\left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{2}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}, \\ \left(\sum_{r=1}^{2}\left(\alpha_r\left(\psi_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{2}\left(\alpha_r\left(\psi_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array}\right).$$

Thus

$$\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda = \left(\begin{array}{c} \left(1-\sum_{r=1}^{2}\left(\alpha_r\left(1-\left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{2}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}, \\ \left(\sum_{r=1}^{2}\left(\alpha_r\left(\psi_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{2}\left(\alpha_r\left(\psi_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array}\right).$$

Let suppose that the result is true for $r = k$.

$$\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda \otimes \left(P_3^{\odot\alpha_3}\right)^\lambda \otimes ... \oplus \left(P_k^{\odot\alpha_k}\right)^\lambda$$

$$= \left(\begin{array}{c} \left(1 - \sum_{r=1}^{k} \left(\alpha_r \left(1 - \left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{k}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}, \\ \left(\sum_{r=1}^{k} \left(\alpha_r \left(\psi_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{k}\left(\alpha_r\left(\psi_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array} \right).$$

We shows that the result is true for $r = k + 1$. *So*

$$\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda \otimes \left(P_3^{\odot\alpha_3}\right)^\lambda \otimes ... \otimes \left(P_k^{\odot\alpha_k}\right)^\lambda \otimes \left(P_{k+1}^{\odot\alpha_{k+1}}\right)^\lambda$$

$$= \left(\begin{array}{c} \left(1 - \sum_{r=1}^{k} \left(\alpha_r \left(1 - \left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{k}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}, \\ \left(\sum_{r=1}^{k} \left(\alpha_r \left(\psi_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{k}\left(\alpha_r\left(\psi_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array} \right)$$

$$\otimes \left(\begin{array}{c} , r^{\frac{1}{q}} \left(\left(\alpha_{k+1}\left(1-\left(\eta_{C_{k+1}}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_{k+1}\left(1-\left(\eta_{C_{k+1}}^{im}(u)\right)^q\right)\right)^{\frac{1}{q}}\right)^\lambda}, \\ r^{\frac{1}{q}} \left(\left(\alpha_{k+1}\right)^{\frac{1}{q}} \psi_{C_{k+1}}(u)\right)^\lambda e^{i2\pi r^{\frac{1}{q}}\left(\left(\alpha_{k+1}\right)^{\frac{1}{q}}\psi_{C_{k+1}}^{im}(u)\right)^\lambda} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - \sum_{r=1}^{k+1} \left(\alpha_r \left(1 - \left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{q}}}, \\ \left(\sum_{r=1}^{k+1} \left(\alpha_r \left(\psi_{C_r}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\psi_{C_r}^{im}(u)\right)^q\right)^\lambda\right)^{\frac{1}{q}}} \end{array} \right)$$

Thus

$$\left(\left(\left(\alpha_1 P_1\right)^\lambda \otimes \left(\alpha_2 P_2\right)^\lambda \otimes \left(\alpha_3 P_3\right)^\lambda \otimes ... \otimes \left(\alpha_k P_k\right)^\lambda\right) \otimes \left(\alpha_{k+1} P_{k+1}\right)^\lambda\right)^{\frac{1}{\lambda}}$$

$$= \left(\begin{array}{c} \left(\left(1 - \sum_{r=1}^{k+1} \left(\alpha_r \left(1 - \left(\eta_{C_r}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1-\sum_{r=1}^{k+1}\left(\alpha_r\left(1-\left(\eta_{C_r}^{im}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(\sum_{r=1}^{k+1} \left(\alpha_r \left(\upsilon_r\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_r\left(\upsilon_r^{im}\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \end{array} \right).$$

Theorem 10 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* Cq*-ROF values. Let* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$. *If* $\left(\eta_{C_1}(u)\, e^{i2\pi\left(\eta_{C_1}^{im}(u)\right)}, \psi_{C_1}(u)\, e^{i2\pi\left(\psi_{C_1}^{im}(u)\right)}\right) = \left(\eta_{C_2}(u)\, e^{i2\pi\left(\eta_{C_2}^{im}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi_{C_2}^{im}(u)\right)}\right) = ...$ $= \left(\eta_{C_m}(u)\, e^{i2\pi\left(\eta_{C_m}^{im}(u)\right)}, \psi_{C_m}(u)\, e^{i2\pi\left(\psi_{C_m}^{im}(u)\right)}\right) = \left(\eta_C(u)\, e^{i2\pi\left(\eta_C^{im}(u)\right)}, \psi_C(u)\, e^{i2\pi\left(\psi_C^{im}(u)\right)}\right)$ *and* $\lambda = 1$ *then* $Cq\text{-}ROFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_C(u)\, e^{i2\pi\left(\eta_C^{im}(u)\right)}, \psi_C(u)\, e^{i2\pi\left(\psi_C^{im}(u)\right)}\right)$.

Proof 10 *Similar to the proof of Theorem 2.*

Theorem 11 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* Cq*-ROF values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$. *Let* $P^- = \left(\eta^-_C(u)\, e^{i2\pi\left(\eta^{im-}_C(u)\right)}, \psi^-_C(u)\, e^{i2\pi\left(\psi^{im-}_C(u)\right)}\right)$ *and let* $P^+ = \left(\eta^+_C(u)\, e^{i2\pi\left(\eta^{im+}_{C_r}(u)\right)}, \psi^+_{C_r}(u)\, e^{i2\pi\left(\psi^{im+}_{C_r}(u)\right)}\right)$*, where* $\eta^-_C(u)\, e^{i2\pi\left(\eta^{im-}_C(u)\right)}, = \min\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta^{im}_{C_1}(u)\right)}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta^{im}_{C_2}(u)\right)}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta^{im}_{C_m}(u)\right)},\right\}$, $\psi^-_C(u)\, e^{i2\pi\left(\psi^{im-}_C(u)\right)} = \max\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi^{im}_{C_1}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi^{im}_{C_2}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi^{im}_{C_m}(u)\right)}\right\}$, $\eta^+_C(u)\, e^{i2\pi\left(\eta^{im+}_{C_r}(u)\right)}, = \max\left\{\eta_{C_1}(u)\, e^{i2\pi\left(\eta^{im}_{C_1}(u)\right)}, \eta_{C_2}(u)\, e^{i2\pi\left(\eta^{im}_{C_2}(u)\right)}, ..., \eta_{C_m}(u)\, e^{i2\pi\left(\eta^{im}_{C_m}(u)\right)},\right\}$ *and* $\psi^+_{C_r}(u)\, e^{i2\pi\left(\psi^{im+}_{C_r}(u)\right)} = \min\left\{\psi_{C_1}(u)\, e^{i2\pi\left(\psi^{im}_{C_1}(u)\right)}, \psi_{C_2}(u)\, e^{i2\pi\left(\psi^{im}_{C_2}(u)\right)}, ..., \psi_{C_m}(u)\, e^{i2\pi\left(\psi^{im}_{C_m}(u)\right)}\right\}$ *and* $\lambda = 1$. *Then* $P^- \leq Cq$*-*$ROFWGM_E(P_1, P_2, P_3, ..., P_m) \leq P^+$.

Proof 11 *Similar to the proof of Theorem 3.*

Theorem 12 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P^*_r = \left(\eta^*_{C_r}(u)\, e^{i2\pi\left(\eta^{im*}_{C_r}(u)\right)}, \psi^*_{C_r}(u)\, e^{i2\pi\left(\psi^{im*}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* Cq*-ROF values. If* $\eta_{C_r}(u) \leq \eta^*_{C_r}(u)$, $\eta^{im}_{C_r}(u) \leq \eta^{im*}_{C_r}(u)$, $\psi_{C_r}(u) \geq \psi^*_{C_r}(u)$ *and* $\psi^{im}_{C_r}(u) \geq \psi^{im*}_{C_r}(u)$ *where* $r = 1, 2, ..., m$, *then* Cq*-*$ROFWGM_E(P_1, P_2, P_3, ..., P_m)$ $\leq Cq - ROFWGM_E(P^*_1, P^*_2, P^*_3, ..., P^*_m)$.

Proof 12 *Similar to the proof of Theorem 4.*

Definition 8 *Let* $\{P_r : r = 1, 2, ..., m\}$ *be the collection of* Cq*-ROF values and let* Cq*-*$ROFOWGM : \Omega^m \to \Omega$, *if* Cq*-*$ROFOWGM_E(P_1, P_2, P_3, ..., P_m) = \left(\left(P^{\odot\alpha_1}_{\delta(1)}\right)^\lambda \otimes \left(P^{\odot\alpha_2}_{\delta(2)}\right)^\lambda \otimes \left(P^{\odot\alpha_3}_{\delta(3)}\right)^\lambda \otimes ... \otimes \left(P^{\odot\alpha_m}_{\delta(r)}\right)^\lambda\right)^{\frac{1}{\lambda}}$ *then* Cq*-*$ROFOWGM_E$ *is called a* q*-rung orthopair fuzzy ordered weighted geometric operator of dimension* n*, where* $(\delta(1), \delta(2), ..., \delta(m))$ *is a permutation of* $(1, 2, ..., m)$ *such that* $P_{\delta(r-1)} \geq P_{\delta(r)}$ *for all* r*,* Ω *is the set of all* Cq*-ROF values,* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\{P_r : r = 1, 2, ..., m\}$ *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$.

Theorem 13 *Let* $\left\{P_r = \left(\eta_{C_r}(u)\, e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)}, \psi_{C_r}(u)\, e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* Cq*-ROF values. Then by using the* Cq*-*$ROFOWGM_E$ *operator their aggregated value is also a* Cq*-ROF value and*

$$\begin{aligned} & Cq - ROFOWGM_E(P_1, P_2, P_3, ..., P_m) \\ & = \begin{pmatrix} \left(\left(1 - \sum\limits_{r=1}^{m}\left(\alpha_r\left(1 - \left(\eta_{C_{\delta(r)}}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(1 - \sum\limits_{r=1}^{m}\left(\alpha_r\left(1 - \left(\eta^{im}_{C_{\delta(r)}}(u)\right)^q\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}}, \\ \left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\psi_{C_{\delta(r)}}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}} e^{i2\pi\left(\left(\sum\limits_{r=1}^{m}\left(\alpha_r\left(\psi^{im}_{C_{\delta(r)}}(u)\right)^q\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{q}}} \end{pmatrix} \end{aligned}$$

$E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{n}\alpha_r=1$.

Proof 13 *Similar to the proof of Theorem 9.*

Theorem 14 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values. Let* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$*. If*
$\left(\eta_{C_1}(u)\,e^{i2\pi\left(\eta^{im}_{C_1}(u)\right)},\psi_{C_1}(u)\,e^{i2\pi\left(\psi^{im}_{C_1}(u)\right)}\right)=\left(\eta_{C_2}(u)\,e^{i2\pi\left(\eta^{im}_{C_2}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi^{im}_{C_2}(u)\right)}\right)=...$
$=\left(\eta_{C_m}(u)\,e^{i2\pi\left(\eta^{im}_{C_m}(u)\right)},\psi_{C_m}(u)\,e^{i2\pi\left(\psi^{im}_{C_m}(u)\right)}\right)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta^{im}_{C}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi^{im}_{C}(u)\right)}\right)$ *and* $\lambda=1$*, then* $Cq\text{-}ROFOWGM_E(P_1,P_2,P_3,...,P_m)=\left(\eta_{C}(u)\,e^{i2\pi\left(\eta^{im}_{C}(u)\right)},\psi_{C}(u)\,e^{i2\pi\left(\psi^{im}_{C}(u)\right)}\right)$.

Proof 14 *Similar to the proof of Theorem 2.*

Theorem 15 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values and* $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$*. Let* $P^-=\left(\eta^-_{C}(u)\,e^{i2\pi\left(\eta^{im-}_{C}(u)\right)},\psi^-_{C}(u)\,e^{i2\pi\left(\psi^{im-}_{C}(u)\right)}\right)$ *and let* $P^+=\left(\eta^+_{C}(u)\,e^{i2\pi\left(\eta^{im+}_{C_r}(u)\right)},\psi^+_{C_r}(u)\,e^{i2\pi\left(\psi^{im+}_{C_r}(u)\right)}\right)$*, where*
$\eta^-_{C}(u)\,e^{i2\pi\left(\eta^{im-}_{C}(u)\right)}=\min\left\{\eta_{C_1}(u)\,e^{i2\pi\left(\eta^{im}_{C_1}(u)\right)},\eta_{C_2}(u)\,e^{i2\pi\left(\eta^{im}_{C_2}(u)\right)},...,\eta_{C_m}(u)\,e^{i2\pi\left(\eta^{im}_{C_m}(u)\right)},\right\}$,
$\psi^-_{C}(u)\,e^{i2\pi\left(\psi^{im-}_{C}(u)\right)}=\max\left\{\psi_{C_1}(u)\,e^{i2\pi\left(\psi^{im}_{C_1}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi^{im}_{C_2}(u)\right)},...,\psi_{C_m}(u)\,e^{i2\pi\left(\psi^{im}_{C_m}(u)\right)}\right\}$,
$\eta^+_{C}(u)\,e^{i2\pi\left(\eta^{im+}_{C_r}(u)\right)},=\max\left\{\eta_{C_1}(u)\,e^{i2\pi\left(\eta^{im}_{C_1}(u)\right)},\eta_{C_2}(u)\,e^{i2\pi\left(\eta^{im}_{C_2}(u)\right)},...,\eta_{C_m}(u)\,e^{i2\pi\left(\eta^{im}_{C_m}(u)\right)},\right\}$ *and*
$\psi^+_{C_r}(u)\,e^{i2\pi\left(\psi^{im+}_{C_r}(u)\right)}=\min\left\{\psi_{C_1}(u)\,e^{i2\pi\left(\psi^{im}_{C_1}(u)\right)},\psi_{C_2}(u)\,e^{i2\pi\left(\psi^{im}_{C_2}(u)\right)},...,\psi_{C_m}(u)\,e^{i2\pi\left(\psi^{im}_{C_m}(u)\right)}\right\}$ *and* $\lambda=1$*. Then* $P^-\leq Cq\text{-}ROFOWGM_E(P_1,P_2,P_3,...,P_m)\leq P^+$.

Proof 15 *Similar to the proof of Theorem 3.*

Theorem 16 *Let* $\left\{P_r=\left(\eta_{C_r}(u)\,e^{i2\pi\left(\eta^{im}_{C_r}(u)\right)},\psi_{C_r}(u)\,e^{i2\pi\left(\psi^{im}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *and* $\left\{P^*_r=\left(\eta^*_{C_r}(u)\,e^{i2\pi\left(\eta^{im*}_{C_r}(u)\right)},\psi^*_{C_r}(u)\,e^{i2\pi\left(\psi^{im*}_{C_r}(u)\right)}\right):r=1,2,...,m\right\}$ *are two collections of* Cq*-ROF values. If* $\eta_{C_r}(u)\leq\eta^*_{C_r}(u),\eta^{im}_{C_r}(u)\leq\eta^{im*}_{C_r}(u),\psi_{C_r}(u)\geq\psi^*_{C_r}(u)$ *and* $\psi^{im}_{C_r}(u)\geq\psi^{im*}_{C_r}(u)$ *where* $r=1,2,...,m$*, then* $Cq\text{-}ROFOWGM_E(P_1,P_2,P_3,...,P_m)$
$\leq Cq\text{-}ROFOWGM_E(P^*_1,P^*_2,P^*_3,...,P^*_m)$.

Proof 16 *Similar to the proof of Theorem 4.*

Definition 9 *Let* $E=\{b_i:i=1,2,3,...,n\}$ *be a collection of* Cq*-ROFP values and* $u_i\in X$*. Then the entropy measure of* E *is defined by*

$$M^q(E^{re})=\frac{1}{n}\sum_{i=1}^{n}\left(1-\sqrt[q]{\frac{2}{3}}\left(\sqrt[q]{\frac{|(\eta_E(u_i))^q-(\psi_i(u_i))^q|}{2-|(\eta_E(u_i))^q-(\psi_i(u_i))^q|}}\right)\left(\sqrt[q]{\frac{2+|-(\eta_E(u_i))^q+(\psi_E(u_i))^q|}{1+|(\eta_E(u_i))^q-(\psi_E(u_i))^q|}}\right)\right)$$

and

$$M^q\left(E^{im}\right)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\sqrt[q]{\frac{2}{3}}\left(\sqrt[q]{\frac{\left|\left(\eta_E^{im}\left(u_i\right)\right)^q-\left(\psi_i^{im}\left(u_i\right)\right)^q\right|}{2-\left|\left(\eta_E^{im}\left(u_i\right)\right)^q-\left(\psi_i^{im}\left(u_i\right)\right)^q\right|}}\right)\left(\sqrt[q]{\frac{2+\left|-\left(\eta_E^{im}\left(u_i\right)\right)^q+\left(\psi_E^{im}\left(u_i\right)\right)^q\right|}{1+\left|\left(\eta_E^{im}\left(u_i\right)\right)^q-\left(\psi_E^{im}\left(u_i\right)\right)^q\right|}}\right)\right)$$

where $0\leq M^q\left(E^{re}\right), M^q\left(E^{im}\right)\leq 1$ *and* $q\geq 1.$

Proposition 1 *Let* $E=\left\{P_r=\left(\eta_{C_r}\left(u\right)e^{i2\pi\left(\eta_{C_r}^{im}(u)\right)},\psi_{C_r}\left(u\right)e^{i2\pi\left(\psi_{C_r}^{im}(u)\right)}\right):r=1,2,...,m\right\}$ *be the collection of* Cq*-ROF values. Then the proposed entropy* $M^q\left(E^{re}\right)$ *for* E *satisfies the following conditions:*

(i) $M^q\left(E^{re}\right)=0$ *if and only if* E *is a crisp set;*

(ii) $M^q\left(E^{re}\right)=1$ *if and only if* $\eta_E\left(u_i\right)=\psi_E\left(u_i\right)$ *for each* $u_i\in X;$

(iii) $M^q\left(E^{re}\right)^c=M^q\left(E^{re}\right);$

(iv) $M^q\left(E_1^{re}\right)\leq M^q\left(E_2^{re}\right)$ *if* E_1^{re} *is less than* E_2^{re}*.*

4. Novel MADM technique using Cq-rung orthopair fuzzy weighted arithmetic/geometric mean aggregation operators

The MADM is a very significant procedure to select an alternative among the set of desirable alternatives in accordance with the opinions of some experts corresponding to attributes. In many real-world situations, MADM is used, and the decision-maker always strives for the best and most rational decision-making outcome. Because of this, choosing the right approach for making decisions is crucial, and multiple strategies should be used for making decisions in various scenarios. Due to this, we suggest a method to address the MADM problem in this part based on the proposed q-rung orthopair fuzzy preference theory.

4.1 Decision-making steps

Let $G=\{G_k:k=1,2,3,...,l\}$ be the set of experts, whose weight vector $w=(w_1,w_2,w_3,...,w_l)$. Let $\mathcal{U}=\{u_i:i=1,2,3,..,m\}$ be the set of alternatives and $C=\{C_t;t=1,2,3,...,n\}$ be the set of attributes. Let $Q^k=((\eta_{it},\psi_{it}))_{m\times n}$ be the decision matrix (Information system) for each expert G_k. In this section, we construct a MADM methodology, including the following key steps:

Step 1 : Compute the q- rung orthopair fuzzy entropy measure of each subset $\left(H_{C_t}\right)^k$ of G_k based on Definition 9 to compute the weight for each C_t of expert G_k, where $\left(H_{C_t}\right)^k=\left\{\left(\left(\eta_{it},\psi_{it}\right)\right)^k:i=1,2,3,...m\right\}$ and ${}_{G_k}\alpha^{C_t}=\frac{M^q\left(\left(H_{C_t}\right)^k\right)}{\sum_{t=1}^{n}\left(M^q\left(\left(H_{C_t}\right)^k\right)\right)}$.

Step 2 : Using Definition 5 or Definition 7, to aggregate the decision matrix G_k, i.e.

$$P_i^k=Cq\text{-}ROFWAM_E\left(P_{i1}^k,P_{i2}^k,P_{i3}^k,...,P_{im}^k\right)$$

or

$$P_i^k=Cq\text{-}ROFWGM_E\left(P_{i1}^k,P_{i2}^k,P_{i3}^k,...,P_{im}^k\right)$$

where $k=1,2,3,...,l.$

Step 3 : Aggregate the difference preferences $P_i^k, i=1,2,3,..,m$ into P_i by applying Definition 5 or Definition 7, i.e.

$$P_i=Cq\text{-}ROFWAM_E\left(P_i^1,P_i^2,P_i^3,...,P_i^l\right)$$

or

$$P_i=Cq\text{-}ROFWGM_E\left(P_i^1,P_i^2,P_i^3,...,P_i^l\right).$$

Step 4 : Compute the score values or accuracy values of the aggregated values obtained in Step 3.

Step 5 : Determine which selection is better by ranking each one based on the Step 4 score values or accuracy values.

Figure 2 presents the flowchart of the proposed model, providing a graphical illustration of the overall methodology.

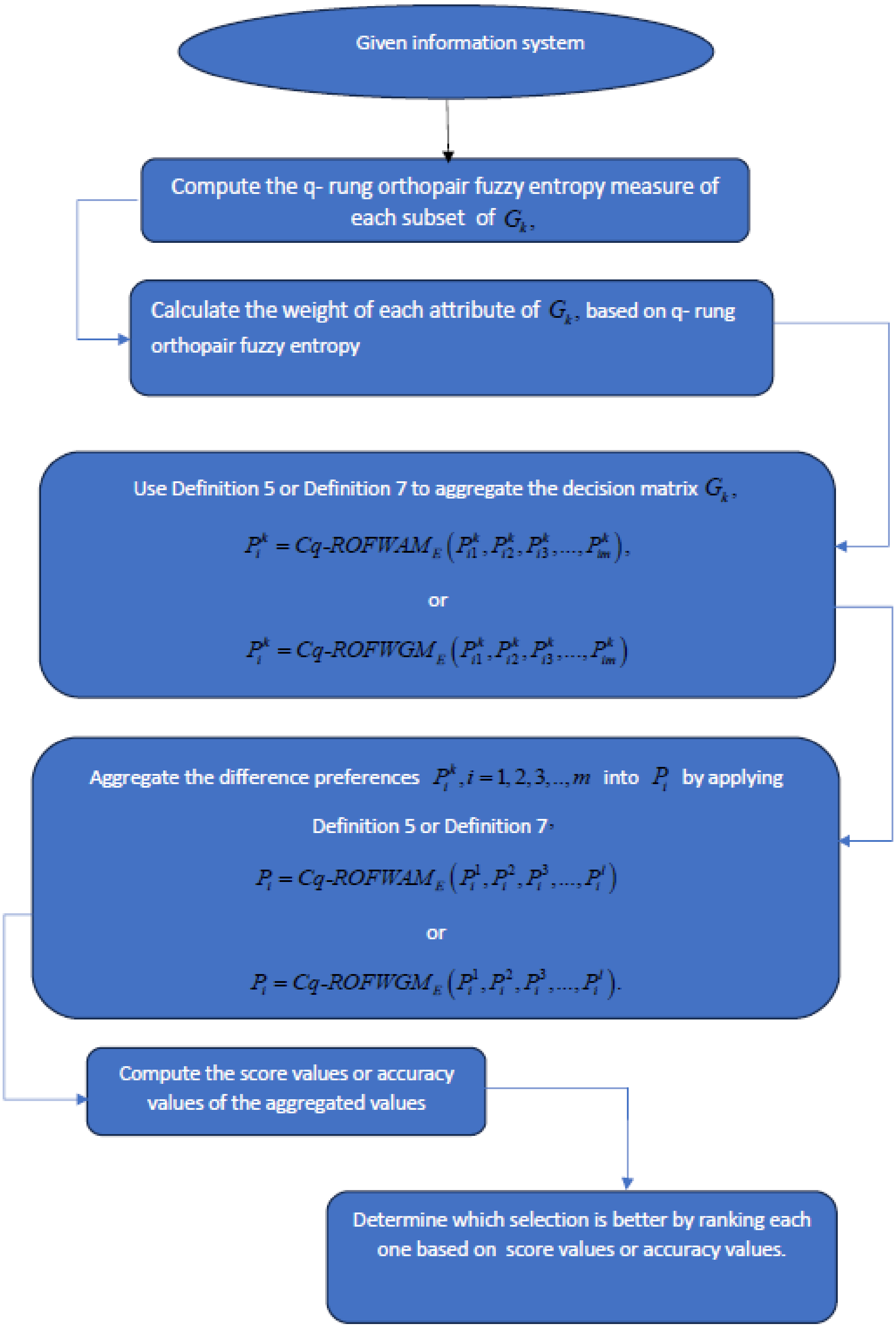

Fig. 2. The flow chart shows the propoded model graphically.

5. Analysis of investments in Chinese education system

Education is a long-term investment, and the highest returns are likely to come from a comprehensive strategy that considers all facets of the educational system. The impact of investments in Chinese education system can also be increased through collaborations with the corporate sector and cooperation with international organizations. The major theme of this problem is finding the best kind of education investment, such as

(1) *Infrastructure development*:

(i) Invest in building and enhancing the infrastructure supporting education, including colleges, institutions, and schools.

(ii) Make sure the buildings have the newest conveniences and technology available to improve the educational environment.

(2) *Teacher training and professional development*:

(i) To raise educational standards, provide funding for thorough teacher training programmes.

(ii) To keep educators abreast of the newest educational techniques and technological advancements, make an investment in continual professional development.

(3) *Online resources for education*:

(i) To support remote and technologically enhanced learning, make investments in digital resources, e-learning platforms, and online libraries.

(ii) Make sure all pupils, regardless of their financial situation, can access educational materials.

(4) *Specialized learning facilities*:

(i) To encourage creativity and competition, set up specialized learning facilities for disciplines like science, technology, engineering, and mathematics (STEM).

(ii) Through inclusive education programs, you may meet the special requirements of pupils who have disabilities.

Figure 3 presents the proposed multi-attribute decision-making (MCDM) framework for evaluating education investment alternatives in the Chinese education system. To find the best alternative among the four alternatives, we consider five attributes, such as

(1) *Risk analysis*:

Like any investment, investing in Chinese education system carries a number of challenges. This is a thorough risk analysis for investments in the education sector in China. It includes risks related to regulations and policies, political unpredictability, economic volatility, security issues, infrastructure, technology, market demand and competition, financial sustainability, currency exchange risks, cultural and social challenges, health risks, demographic trends, and environmental sustainability. Risk analyses and thorough investigations should be carried out prior to making any investments in Chinese educational system. The best ways to reduce these risks are to interact with local stakeholders, comprehend the regulatory environment, and be flexible in the face of changing conditions. Resistance to different issues can also be improved by diversifying investments and putting backup plans in place.

(2) *Growth condition*:

Diverse economic circumstances can impact investments in the education sector of China. Making wise investment choices requires a comprehension of the elements that support the expansion of the educational industry. The following circumstances for expansion can have a beneficial influence on investments in Chinese education sector: international relationships and investments; support from the government and policies; increasing population and youth demographic information; financial stability; innovations in technology; industry alignment and vocational training; quality control and authorization; entrepreneurial spirit and innovation; building new infrastructure; public-private relationships; and globalization of education. Prior to adopting any judgments on their investments, investors in the educational sector should carefully consider these growth factors and carry out indepth market analysis. Factors for achievement in the field include utilizing technological advances, keeping an eye on relevance and effectiveness, and adjusting to shifting educational demands.

(3) *Social political impact*:

Investments in the education industry may be significantly impacted by Chinese political and social climate. Stakeholders and investors alike need to understand the dynamics of this ecosystem. The following significant social and political variables may have an impact on developments in Pakistan's education sector: Social unrest and safety issues, the urban-rural divide, gender inequalities in youth demographics, involvement in the community, government strategies and objectives, regulatory atmosphere, public investment and allocation of funds, stability in politics, decentralization, and governance at the local level, and inclusive education policies are some of the factors that influence educational views in society. It is recommended that investors keep a careful eye on Chinese changing social and political scene, actively participate in local communities, and remain up to date on the regulations and intentions of the governing body. Long-term achievement in the education sector depends on fostering resilience to future political and social obstacles and coordinating resources with the nation's larger aspirations for advancement in society.

(4) *Environmental impact*:

Despite the reality that environmental effects on investments in educational institutions in China may not be as direct as in other sectors, a number of environmental factors can still affect how decisions are made. These include sustainable development, awareness of the environment, resiliency to climate change, renewable energy sources, biodiversity conservation, disposal of waste, transportation, and air quality, community green initiatives, eco-friendly practices, research in the field of environmental sciences, and adapt. Even though they might not be the main factors influencing investment choices in the field of education, these factors can nonetheless have an impact on the image and long-term viability of educational institutions. Aligning strategy with awareness of the environment should be beneficial for investors and educational stakeholders, resulting in a stronger and more sustainable educational system in Pakistan.

(5) *Development of society*:

In China, the growth of society is a major factor in creating an atmosphere that is favorable to investments in the education sector. Higher expectations for education, higher-quality educational institutions, and a workforce with greater skill and education can all be attributed to a developed society. The following significant facets of societal growth have the potential to increase investment in the education sector: The following are some examples of social accountability and awareness in education: community engagement, skill development, continuous learning, cultural acceptance of education, tolerance, and variation, awareness of the environment, charity, and ethical behavior. The aforementioned elements are related to one another, and societal evolution is a complicated and multidimensional process. Investments in education that support and advance social development objectives are probably going to have a bigger, longer-lasting effect on Pakistan's educational system.

Let $E = \{G_1,\ G_2,\ G_3\}$ be the set of three experts. Let $\mathcal{U} = \{u_1,\ u_2,\ u_3,\ u_4\}$ be the set of alternatives, where u_1 : infrastructure development, u_2 : teacher training and professional development, u_3 : online resources for education, u_4 : specialized learning facilities, and $C = \{C_1,\ C_2,\ C_3,\ C_4,\ C_5\}$ be the set of attributes, where there attributes are risk analysis, growth condition, social political impact, environmental impact and development of society. Based on these attributes, we evaluated the best kind of alternatives among our four alternatives. For this, we consider the following weight vector for our considered decision makers G_1, G_2 and G_3, such as $(0.3, 0.3, 0.4)^T$. Thus, we considered our procedure and tried to evaluate our targeted result. Tables $1, 2$ and 3 represents the information table given by experts G_1, G_2 and G_3.

Based on Step 1, let $q = 3$ and applying q-rung orthopair fuzzy entropy measure for each subset $(H_{C_t})^1$ of G_1 based on Definition 9 to compute the weight for each C_t of expert $G_1, t = 1, 2, 3, 4, 5.$

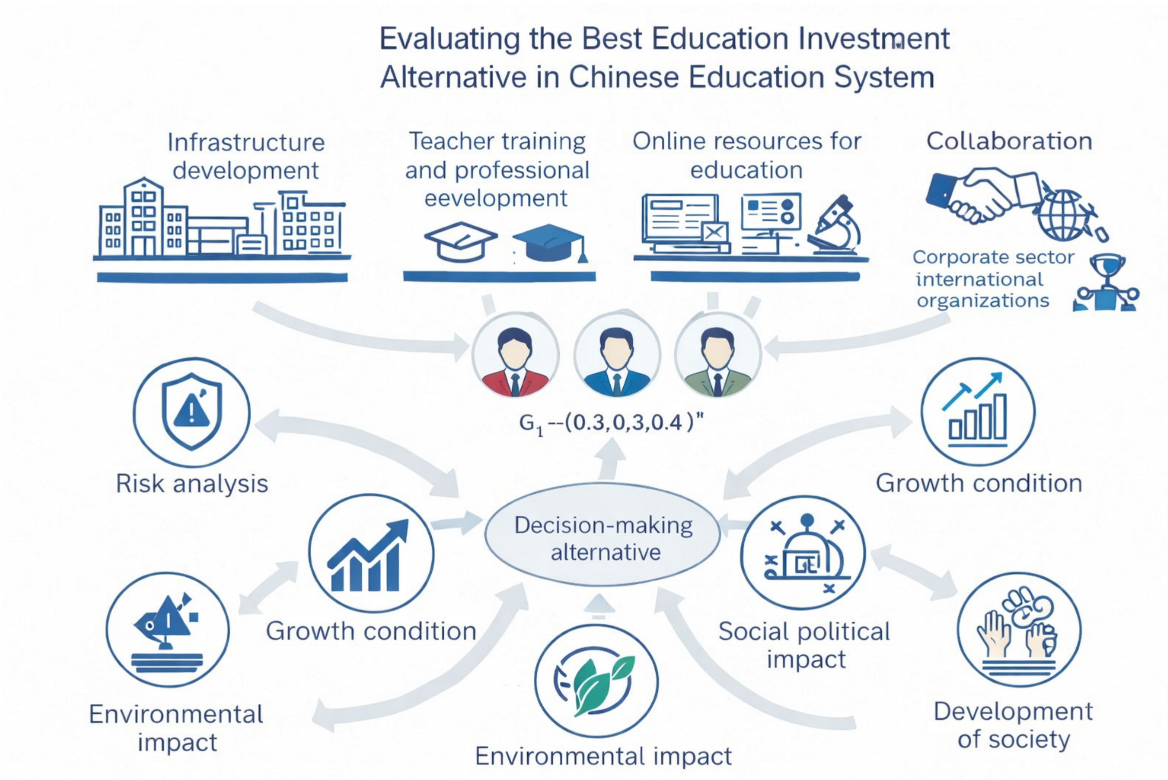

Fig. 3. Multi-Attribute Decision-Making Framework for Evaluating Education Investment Alternatives in the Chinese Education System.

Table 1
Information table given by expert G_1

	C_1	C_2	C_3	C_4	C_5
u_1	$\left(0.70e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.500\right)$	$\left(0.30e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$
u_2	$\left(0.50e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.70e^{i2\pi(0)}, 0.80e^{i2\pi(0)}\right)$	$\left(0.70e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$	$\left(0.70e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$
u_3	$\left(0.40e^{i2\pi(0)}, 0.90e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.90e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$
u_4	$\left(0.80e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.80e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$	$\left(0.80e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$

Table 2
Information table given by expert G_2

	C_1	C_2	C_3	C_4	C_5
u_1	$\left(0.40e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.90e^{i2\pi(0)}\right)$
u_2	$\left(0.50e^{i2\pi(0)}, 0.90e^{i2\pi(0)}\right)$	$\left(0.60e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.80e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$
u_3	$\left(0.60e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.80e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.30e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.70e^{i2\pi(0)}, 0.40e^{i2\pi(0)}\right)$
u_4	$\left(0.80e^{i2\pi(0)}, 0.60e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.90e^{i2\pi(0)}\right)$	$\left(0.30e^{i2\pi(0)}, 0.70e^{i2\pi(0)}\right)$	$\left(0.40e^{i2\pi(0)}, 0.50e^{i2\pi(0)}\right)$	$\left(0.50e^{i2\pi(0)}, 0.80e^{i2\pi(0)}\right)$

Table 3
Information table given by expert G_3

	C_1	C_2	C_3	C_4	C_5
u_1	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.40e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.50e^{i2\pi(0)}, \\ 0.90e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.30e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0)}, \\ 0.40e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0)}, \\ 0.50e^{i2\pi(0)} \end{pmatrix}$
u_2	$\begin{pmatrix} 0.30e^{i2\pi(0)}, \\ 0.60e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.70e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0)}, \\ 0.50e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.40e^{i2\pi(0)}, \\ 0.60e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.50e^{i2\pi(0)}, \\ 0.90e^{i2\pi(0)} \end{pmatrix}$.
u_3	$\begin{pmatrix} 0.40e^{i2\pi(0)}, \\ 0.50e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0)}, \\ 0.50e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.40e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.70e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0)}, \\ 0.70e^{i2\pi(0)} \end{pmatrix}$
u_4	$\begin{pmatrix} 0.80e^{i2\pi(0)}, \\ 0.70e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0)}, \\ 0.40e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.50e^{i2\pi(0)}, \\ 0.60e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.50e^{i2\pi(0)}, \\ 0.90e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0)}, \\ 0.40e^{i2\pi(0)} \end{pmatrix}$

Table 4 represents the weights for each ${}_{G_1}\alpha^{C_t}$, where

Table 4
Weights assigned by expert G_1

Expert	${}_{G_1}\alpha^{C_1}$	${}_{G_1}\alpha^{C_2}$	${}_{G_1}\alpha^{C_3}$	${}_{G_1}\alpha^{C_4}$	${}_{G_1}\alpha^{C_5}$
G_1	0.16816	0.21706	0.17460	0.23592	0.20426

Applying q-rung orthopair fuzzy entropy measure for each subset $(H_{C_t})^2$ of G_2 based on Definition 9 to compute the weights weight for each C_t of expert $G_2, t = 1, 2, 3, 4, 5$ and $q = 3$. Table 5 represents the weights for each ${}_{G_2}\alpha^{C_t}$, where

Table 5
Weights assigned by expert G_2

	${}_{G_2}\alpha^{C_1}$	${}_{G_2}\alpha^{C_2}$	${}_{G_2}\alpha^{C_3}$	${}_{G_2}\alpha^{C_4}$	${}_{G_2}\alpha^{C_5}$
G_2	0.20947	0.19875	0.19984	0.22769	0.16425

Further, we apply q-rung orthopair fuzzy entropy measure for each subset $(H_{C_t})^3$ of G_2 based on Definition 9 to compute the weights weight for each C_t of expert G_3, $t = 1, 2, 3, 4, 5$, where $q = 3$. Table 6 represents the weights for each ${}_{G_3}\alpha^{C_t}$, where Based on Step 2, the aggregated values obatain

Table 6
Weights assigned by expert G_3

	${}_{G_3}\alpha^{C_1}$	${}_{G_3}\alpha^{C_2}$	${}_{G_3}\alpha^{C_3}$	${}_{G_3}\alpha^{C_4}$	${}_{G_3}\alpha^{C_5}$
G_3	0.24266	0.17281	0.19963	0.17961	0.20529

from expert G_1 by using Cq-$ROFWAM$ operator with $q = 3$ and $\lambda = 1$. Table 7 represent aggregated values obtain from expert G_1, we have Utilizing Step 2, the aggregated values obatain form expert

Table 7
Aggregated values obtained from expert G_1

	u_1	u_2	u_3	u_4
	$\begin{pmatrix} 0.53459e^{i2\pi(0)}, \\ 0.567685e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.629419e^{i2\pi(0)}, \\ 0.650878e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.498457e^{i2\pi(0)}, \\ 0.730804e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.662366e^{i2\pi(0)}, \\ 0.623799e^{i2\pi(0)} \end{pmatrix}$

G_2 by using Cq-$ROFWAM$ operator with $q = 3$ and $\lambda = 1$. Table 8 represent aggregated values obtain from expert G_2, we have Based on Step 2, the aggregated values obatain from expert G_3 by using Cq-$ROFWAM$ operator with $q = 3$ and $\lambda = 1$. Table 9 represent aggregated values obtain from expert G_3, we have

Table 8
Aggregated values obtained from expert G_2

	u_1	u_2	u_3	u_4
	$\begin{pmatrix} 0.520619e^{i2\pi(0)}, \\ 0.67312e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.523041e^{i2\pi(0)}, \\ 0.741011e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.61701e^{i2\pi(0)}, \\ 0.556103e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.543415e^{i2\pi(0)}, \\ 0.718707e^{i2\pi(0)} \end{pmatrix}$

Table 9
Aggregated values obtained from expert G_3

	u_1	u_2	u_3	u_4
	$\begin{pmatrix} 0.653835e^{i2\pi(0)}, \\ 0.568833e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.530751e^{i2\pi(0)}, \\ 0.687604e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.58584e^{i2\pi(0)}, \\ 0.581603e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.742766e^{i2\pi(0)}, \\ 0.655372e^{i2\pi(0)} \end{pmatrix}$

Furthermore, applying Step 3, the aggregated values obatain from Tables $7, 8$ and 9 by using q-$ROFWAM$ operator with $q = 3$. Table 10 shows the aggregated values obatain from Tables $7, 8$ and 9. Applying Definition 3 in accordance with Step 4 to determine the score values of the aggregated values in Table 10 are

Table 10
Aggregated values obtained from Tables 7, 8, and 9

	u_1	u_2	u_3	u_4
	$\begin{pmatrix} 0.584780e^{i2\pi(0)}, \\ 0.603686e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.562039e^{i2\pi(0)}, \\ 0.694398e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.572918e^{i2\pi(0)}, \\ 0.627753e^{i2\pi(0)} \end{pmatrix}$	$\begin{pmatrix} 0.668866e^{i2\pi(0)}, \\ 0.667039e^{i2\pi(0)} \end{pmatrix}$

$$\Psi(u_1) = -0.02003,\ \Psi(u_2) = -0.15729,\ \Psi(u_3) - 0.05933 \text{ and } \Psi(u_4) = 0.00245.$$

Using the score values acquired above, the alternatives are ranked based on Step 5, we get

$$u_4 \succ u_1 \succ u_3 \succ u_2.$$

Let $E = \{G_1\}$ be the set of one expert. Let $\mathcal{U} = \{u_1,\ u_2,\ u_3,\ u_4\}$ be the set of alternatives, where u_1 : infrastructure development, u_2 : teacher training and professional development, u_3 : online resources for education, u_4 : specialized learning facilities, and $C = \{C_1,\ C_2,\ C_3,\ C_4\}$ be the set of attributes, where there attributes are growth condition, social political impact, environmental impact and development of society. Based on these attributes, we evaluated the best kind of alternatives among our four alternatives. For this, we consider the following weight vector for our considered attributes C_1, C_2, C_3, and C_4 such as $(0.3471, 0.2316, 0.2308, 0.1905)$. Thus, we considered our procedure and tried to evaluate our targeted result. Table 11 represent the information table given by expert G_1.

Table 11
Information table given by expert G_1

	C_1	C_2	C_3	C_4
u_1	$\begin{pmatrix} 0.6095e^{i2\pi(0.2967)}, \\ 0.1179e^{i2\pi(0.1960)}, \end{pmatrix}$	$\begin{pmatrix} 0.5820e^{i2\pi(0.3216)}, \\ 0.4278e^{i2\pi(0.5262)} \end{pmatrix}$	$\begin{pmatrix} 0.3856e^{i2\pi(0.4019)}, \\ 0.8138e^{i2\pi(0.6079)} \end{pmatrix}$	$\begin{pmatrix} 0.5897e^{i2\pi(0.5132)}, \\ 0.2632e^{i2\pi(0.07280)} \end{pmatrix}$
u_2	$\begin{pmatrix} 0.6095e^{i2\pi(0.4733)}, \\ 0.2347e^{i2\pi(0.09798)} \end{pmatrix}$	$\begin{pmatrix} 0.7823e^{i2\pi(0.6479)}, \\ 0.2251e^{i2\pi(0.1000)} \end{pmatrix}$	$\begin{pmatrix} 0.8855e^{i2\pi(0.3291)}, \\ 0.3291e^{i2\pi(0.4455)} \end{pmatrix}$	$\begin{pmatrix} 0.3995e^{i2\pi(0.4328)}, \\ 0.2632e^{i2\pi(0.1459)} \end{pmatrix}$
u_3	$\begin{pmatrix} 0.4919e^{i2\pi(0.7764)}, \\ 0.3503e^{i2\pi(0.3883))} \end{pmatrix}$	$\begin{pmatrix} 0.6812e^{i2\pi(0.4836)}, \\ 0.3277e^{i2\pi(0.3612)} \end{pmatrix}$	$\begin{pmatrix} 0.5811e^{i2\pi(0.5647)}, \\ 0.5273e^{i2\pi(0.6877)} \end{pmatrix}$	$\begin{pmatrix} 0.7620e^{i2\pi(0.8031)}, \\ 0.3527e^{i2\pi(0.2191)} \end{pmatrix}$
u_4	$\begin{pmatrix} 0.8639e^{i2\pi(0.5695)}, \\ 0.4637e^{i2\pi(0.2927)} \end{pmatrix}$	$\begin{pmatrix} 0.4836e^{i2\pi(0.3216)}, \\ 0.3277e^{i2\pi(0.4444)} \end{pmatrix}$	$\begin{pmatrix} 0.3856e^{i2\pi(0.4019)}, \\ 0.6239e^{i2\pi(0.5275)} \end{pmatrix}$	$\begin{pmatrix} 0.7620e^{i2\pi(0.3477)}, \\ 0.1749e^{i2\pi(0.2929)} \end{pmatrix}$

Table 12
Aggregated values obtained from expert G_1

u_1	$\left(0.554937e^{i2\pi(0.376778)}, 0.4618e^{i2\pi(0.404674)}\right)$
u_2	$\left(0.694304e^{i2\pi(0.485444)}, 0.262771e^{i2\pi(0.235606)}\right)$
u_3	$\left(0.616445e^{i2\pi(0.67813)}, 0.394124e^{i2\pi(0.448161)}\right)$
u_4	$\left(0.676862e^{i2\pi(0.443665)}, 0.441781e^{i2\pi(0.39502)}\right)$

Utilizing Step $2,$ the aggregated values obatain form expert G_1 by using Cq-$ROFWAM$ operator with $q = 2$ and $\lambda = 1.$ Table 12 represent aggregated values obatain from expert $G_1,$ we have Applying Definition 3 in accordance with Step 4 to determine the score values of the aggregated values in Table 12 are

$$\Psi\left(u_1\right) = 0.03645,\ \Psi\left(u_2\right) = 0.29658,\ \Psi\left(u_3\right) = 0.24184 \text{ and } \Psi\left(u_4\right) = 0.15189.$$

Using the score values acquired above, the alternatives are ranked based on Step $5,$ we get

$$u_2 \succ u_3 \succ u_4 \succ u_1.$$

Similarly based on Step $2,$ the aggregated values obatain form expert G_1 by using Cq-$ROFWGM$ operator with $q = 2$ and $\lambda = 1.$ Table 13 represent aggregated values obatain from expert $G_1,$ we have Applying Definition 3 in accordance with Step 4 to determine the score values of the aggregated values in Table 13 are

Table 13
Aggregated values obtained from expert G_1

u_1	$\left(0.554937e^{i2\pi(0.376778)}, 0.4618e^{i2\pi(0.404674)}\right)$
u_2	$\left(0.694304e^{i2\pi(0.485444)}, 0.262771e^{i2\pi(0.235606)}\right)$
u_3	$\left(0.616445e^{i2\pi(0.67813)}, 0.394124e^{i2\pi(0.448161)}\right)$
u_4	$\left(0.676862e^{i2\pi(0.443665)}, 0.441781e^{i2\pi(0.39502)}\right)$

$$\Psi\left(u_1\right) = 0.03645,\ \Psi\left(u_2\right) = 0.29658,\ \Psi\left(u_3\right) = 0.24184 \text{ and } \Psi\left(u_4\right) = 0.15189.$$

Using the score values acquired above, the alternatives are ranked based on Step $5,$ we get

$$u_2 \succ u_3 \succ u_4 \succ u_1.$$

6. Comparative and Sensitive Analysis

6.1 Comparative analysis for q-rung orthopair fuzzy enviornment

Based on the data presented in Tables 1, 2, 3, 14 and 15, we compare the proposed Cq-ROFWAM and Cq-ROFWGM methods with several existing aggregation operators. The comparative results demonstrate that the proposed methods and the existing approaches identify the same optimal alternative, thereby confirming the reliability and effectiveness of the proposed techniques. For comparison, several representative aggregation operators available in the literature are considered. Xu *et al.,* [19] proposed the q-rung orthopair fuzzy dual hesitant Heronian mean operators and demonstrated their applications in multiple-attribute group decision-making. Wei *et al.,* [20] developed q-rung orthopair fuzzy Heronian mean aggregation operators for multi-attribute decision-making. Yager *et al.,* [21] investigated the properties of generalized orthopair fuzzy sets and discussed their practical applications. Yang and Pang [22] established a theoretical framework for partitioned Bonferroni mean operators under the q-rung orthopair fuzzy environment. Ye *et al.,* [23] investigated single-variable differential calculus in the q-rung orthopair fuzzy environment and demonstrated its applications. Peng *et al.,* [32] introduced exponential operational laws and aggregation operators for q-rung orthopair fuzzy sets and proposed a novel score function for decision-making. Garg and Chen [33] introduced neutrality aggregation operators for q-rung orthopair fuzzy sets and applied them to multi-attribute group decision-making problems. Using the information presented in Tables 1, 2, 3, 14 and 15, a detailed comparative analysis is performed, and the corresponding results are reported in Table 14. The methods proposed by Peng *et al.,* [32], Ali [34], Wei *et al.,* [20], Xu *et al.,* [19], Yager *et al.,* [21], Yang and Pang [22], Xing *et al.,* [35], Ye *et al.,* [23], and Garg and Chen [33], together with the proposed approach, all identify the same optimal alternative. Furthermore, the ranking results produced by the proposed methods are consistent with those obtained by Xu *et al.,* [19], Peng *et al.,* [32], Ali [34], Wei *et al.,* [20], and Xing *et al.,* [35]. Compared with the existing approaches, the proposed alternative selection methodology based on the Cq-ROFWAM and Cq-ROFWGM operators combined with the entropy measure is more flexible, efficient, and capable of handling complex q-rung orthopair fuzzy information. According to all comparison methods, the alternative u_4 is identified as the optimal choice, as shown in Table 14 and 15 . Although all methods reach the same final decision, their computational procedures differ considerably. The existing methods aggregate information using conventional q-rung orthopair fuzzy weighted averaging and weighted geometric aggregation operators, whereas the proposed approach employs the Cq-ROFWAM and Cq-ROFWGM operators together with the entropy measure to aggregate complex q-rung orthopair fuzzy information. Moreover, different values of λ are considered to investigate the stability of the proposed approach. The ranking results of the alternatives u_i $(i = 1, 2, 3, 4)$ obtained by the Cq-ROFWAM operator for $1 \leq \lambda \leq 50$ are presented in Table 15. Finally, the graphical representation of the comparative results reported in Table 14 is presented in Table 16.

Table 14
Ranking of the alternatives using different methods

Measure	$\Psi(u_1)$	$\Psi(u_2)$	$\Psi(u_3)$	$\Psi(u_4)$	Ranking
Xu *et al.,* [19]	0.14800	0.11000	0.12100	0.10800	$u_4 \succ u_2 \succ u_3 \succ u_1$
Garg and Chen [33]	0.63100	0.64700	0.65300	0.61000	$u_4 \succ u_1 \succ u_2 \succ u_3$
Wei *et al.,* [20]	0.58300	0.63900	0.59600	0.57300	$u_4 \succ u_1 \succ u_3 \succ u_2$
Yager *et al.,* [21]	0.12600	0.14200	0.10300	0.17400	$u_4 \succ u_2 \succ u_1 \succ u_3$
Yang and Pang [22]	0.76100	0.79900	0.75400	0.73100	$u_4 \succ u_3 \succ u_1 \succ u_2$
Ye *et al.,* [23]	0.04420	0.45400	0.42000	0.40600	$u_4 \succ u_3 \succ u_1 \succ u_2$
Ali [34]	0.37200	0.42100	0.40200	0.35800	$u_4 \succ u_1 \succ u_3 \succ u_2$
Peng *et al.,* [32]	0.85300	0.90700	0.87500	0.83300	$u_4 \succ u_1 \succ u_3 \succ u_2$
Lei and Xu [36]	0.81500	0.79400	0.76300	0.74300	$u_4 \succ u_3 \succ u_2 \succ u_1$
Xing *et al.,* [35]	0.23400	0.26500	0.24500	0.22700	$u_4 \succ u_1 \succ u_3 \succ u_2$
Proposed Cq-ROFWAM	-0.02003	-0.15729	-0.05933	0.00245	$u_4 \succ u_1 \succ u_3 \succ u_2$
Proposed Cq-ROFWGM	-0.020031	-0.15730	-0.059333	0.068964	$u_4 \succ u_1 \succ u_3 \succ u_2$

6.2 Comparative analysis for complex q-rung orthopair fuzzy environment

To further evaluate the effectiveness and reliability of the proposed aggregation operators, a comparative analysis is conducted with several representative aggregation methods available in the literature. Specifically, the approaches proposed by Liu *et al.,* [31], Janani *et al.,* [37], Senapati *et al.,* [38], Liu and Wang [39], Seikh and Mandal [40], and Ali and Naeem [41] are selected because they represent important developments in complex fuzzy aggregation and multiple attribute decision-making. Using the decision information presented in Tables 14–16, the proposed Cq-ROFWAM and Cq-ROFWGM operators are applied and the obtained results are compared with those generated by the existing

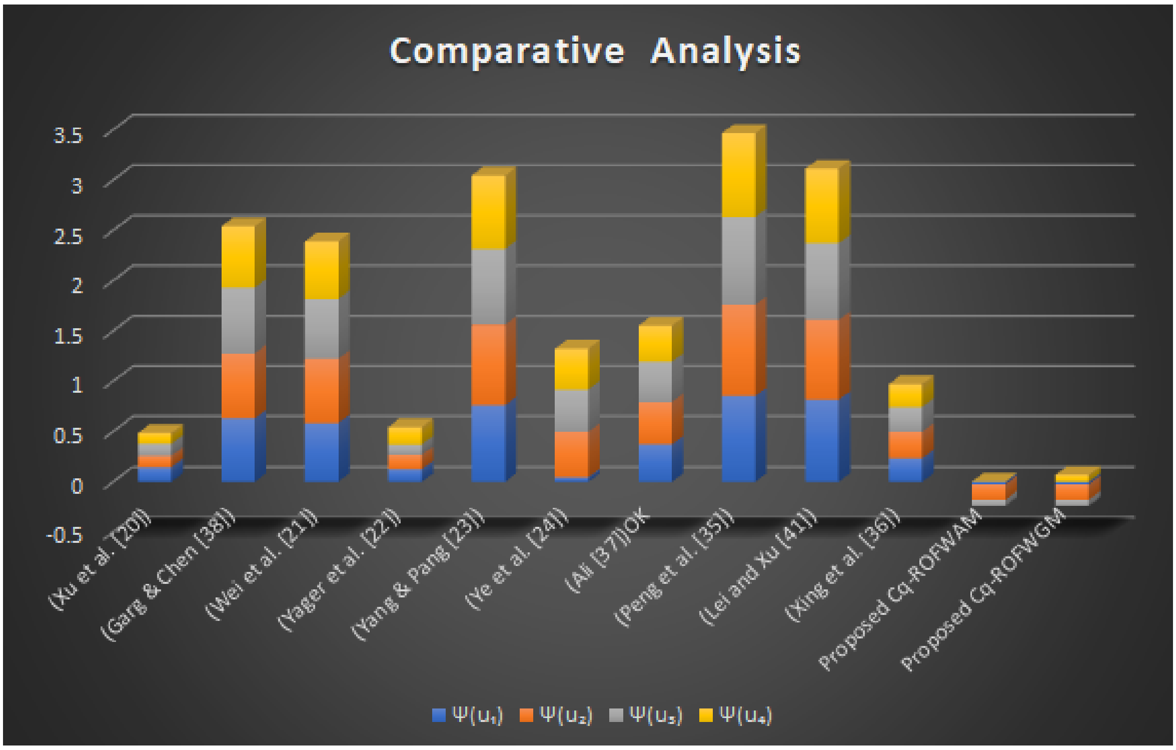

Fig. 4. Table 14 Graphical representation.

methods. The comparative ranking results are summarized in Table 16, while their graphical illustration is presented in Figure 4. It can be observed that all applicable methods identify the same optimal alternative, indicating that the proposed operators preserve the fundamental preference structure of the decision problem. This agreement with existing approaches demonstrates that the proposed aggregation framework is mathematically consistent and produces reliable ranking results comparable with well-established methods in the literature. More importantly, the comparative analysis provides insight into the reason behind the obtained rankings rather than merely confirming their consistency. The highest-ranked alternative (u_4 in this case) achieves superior performance because it exhibits comparatively stronger membership information and lower non-membership information across the most influential attributes. During the aggregation process, these favorable evaluations are consistently reinforced by the proposed operational laws, resulting in a higher overall score. Alternatives with lower rankings contain weaker evaluations or larger uncertainty in one or more important attributes, which decreases their aggregated assessment values. These observations are consistent with previous fuzzy decision-making studies, where the optimal alternative is determined not only by excellent performance in a single attribute but by maintaining balanced and consistently high evaluations across all weighted attributes. Therefore, the proposed operators successfully preserve the comprehensive evaluation behavior reported in earlier aggregation models while providing a more flexible aggregation mechanism. Furthermore, although the proposed method reaches the same decision as several existing approaches, the aggregated score values are not identical. This difference arises from the newly developed algebraic operational laws and the parameterized aggregation mechanism incorporated into the proposed Cq-ROF framework. Unlike conventional operators, the proposed model processes both amplitude and phase information simultaneously, thereby preserving more information during aggregation and reducing information loss. Consequently, the proposed operators provide more discriminative aggregated values while maintaining stable ranking results. The following observations can be drawn from the comparative analysis.

1. All applicable aggregation operators identify the same optimal alternative, demonstrating the

Table 15
Ranking order of alternatives for different λ using Cq-ROFWAM

Measure	$\Psi(u_1)$	$\Psi(u_2)$	$\Psi(u_3)$	$\Psi(u_4)$	Ranking
$\lambda = 1$	-0.02003	-0.15729	-0.05933	0.00245	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 2$	-0.93552	-0.94507	-0.94197	-0.891363	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 3$	-0.9532	-0.9607	-0.95901	-0.92222	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 4$	-0.95845	-0.96578	-0.96431	-0.93066	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 5$	-0.96064	-0.96816	-0.9666	-0.93448	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 10$	-0.96293	-0.97156	-0.96903	-0.93919	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 15$	-0.96315	-0.97224	-0.969244	-0.93987	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 25$	-0.96319	-0.97254	-0.969285	-0.94009	$u_4 \succ u_1 \succ u_3 \succ u_2$
$\lambda = 50$	-0.9632	-0.97259	-0.96929	-0.94011	$u_4 \succ u_1 \succ u_3 \succ u_2$

robustness and consistency of the proposed decision-making framework. The method of Senapati *et al.,* [38] cannot be directly applied because the available information does not satisfy the requirements of intuitionistic fuzzy sets. The remaining methods, however, confirm that the proposed approach preserves the underlying preference ordering of the alternatives.

2. The proposed Cq-ROFWAM and Cq-ROFWGM operators produce score values that are close to those obtained by the methods of Liu and Wang [39], Liu *et al.,* [31], Janani *et al.,* [37], and Ali and Naeem [41]. Nevertheless, the proposed framework employs newly developed algebraic operational laws together with adjustable parameters, allowing greater flexibility during information fusion while preserving the complete complex-valued information.
3. The superior performance of the selected optimal alternative is mainly attributed to its consistently favorable evaluations over the majority of weighted attributes. Since the proposed aggregation operators simultaneously consider both the membership and non-membership information together with their complex-valued characteristics, alternatives exhibiting balanced and strong performances naturally obtain higher aggregated scores. This interpretation explains why the proposed method selects the same best alternative as existing methods while producing more informative score values.
4. Compared with conventional aggregation approaches, the proposed operators reduce information loss by preserving both dimensions of complex q-rung orthopair fuzzy information. Moreover, the adjustable parameter enables decision-makers to incorporate different preference attitudes into the aggregation process without affecting the overall stability of the ranking. This additional flexibility makes the proposed operators more suitable for practical multiple attribute decision-making problems involving complex and uncertain information.
5. Overall, the comparative study demonstrates that the proposed aggregation operators not only reproduce reliable ranking results consistent with previous studies but also provide a richer interpretation of the decision information through their enhanced aggregation mechanism. Therefore, the proposed Cq-ROF framework offers both theoretical improvements and practical advantages over existing aggregation methods. The comparative ranking results are graphically illustrated in Figure 5.

6.3 Future Research Directions

Although the proposed Complex q-Rung Orthopair Fuzzy (Cq-ROF) aggregation framework demonstrates strong capability for managing complex-valued uncertainty in multi-attribute group decision-making, several promising research directions remain open for further investigation.

1. *Extension to Advanced Complex Fuzzy Models:* Future studies may extend the proposed aggregation framework to more expressive uncertainty environments, including complex interval-valued q-rung orthopair fuzzy sets, complex T-spherical fuzzy sets, complex picture fuzzy sets,

Table 16
Ranking of the alternatives using different methods

Measure	$\Psi(u_1)$	$\Psi(u_2)$	$\Psi(u_3)$	$\Psi(u_4)$	Ranking
Liu *et al.,* [31]	0.4142	0.5990	0.4558	0.4220	$u_2 \succ u_3 \succ u_4 \succ u_1$
Liu *et al.,* [31]	0.3616	0.4448	0.3828	0.3487	$u_2 \succ u_3 \succ u_1 \succ u_4$
Janani *et al.,* [37]	0.4338	0.6137	0.4949	0.4499	$u_2 \succ u_3 \succ u_4 \succ u_1$
Janani *et al.,* [37]	0.3816	0.4807	0.4213	0.3863	$u_2 \succ u_3 \succ u_4 \succ u_1$
Janani *et al.,* [37]	0.4248	0.6066	0.4297	0.4341	$u_2 \succ u_4 \succ u_3 \succ u_1$
Janani *et al.,* [37]	0.3805	0.4639	0.3926	0.3719	$u_2 \succ u_3 \succ u_1 \succ u_4$
Senapati *et al.,* [38]	...	...	...	...	Failed
Senapati *et al.,* [38]	...	...	...	...	Failed
Liu and Wang [39]	-0.2016	0.3447	0.02279	-0.2064	$u_2 \succ u_3 \succ u_1 \succ u_4$
Liu and Wang [39]	-0.3654	-0.06875	-0.1055	-0.4275	$u_2 \succ u_3 \succ u_1 \succ u_4$
Seikh and Mandal [40]	-0.2202	0.3111	-0.01570	-0.2364	$u_2 \succ u_3 \succ u_1 \succ u_4$
Seikh and Mandal [40]	-0.3787	-0.06477	-0.1399	-0.4390	$u_2 \succ u_3 \succ u_1 \succ u_4$
Seikh and Mandal [40]	0.1013	0.2629	-0.02599	-0.0276	$u_2 \succ u_3 \succ u_1 \succ u_4$
Seikh and Mandal [40]	-0.4217	-0.1452	-0.1615	-0.4882	$u_2 \succ u_3 \succ u_1 \succ u_4$
Ali and Naeem [41]	0.4248	0.5684	0.4804	0.4341	$u_2 \succ u_3 \succ u_4 \succ u_1$
Ali and Naeem [41]	0.3805	0.4686	0.3926	0.3719	$u_2 \succ u_3 \succ u_1 \succ u_4$
Proposed Cq-ROFWA ($\lambda = 1$)	0.03645	0.29658	0.24184	0.15189	$u_2 \succ u_3 \succ u_4 \succ u_1$
Proposed Cq-ROFWA ($\lambda = 2$)	-0.77708	-0.67620	-0.67812	-0.69832	$u_2 \succ u_3 \succ u_4 \succ u_1$
Proposed Cq-ROFWG ($\lambda = 1$)	0.03645	0.29658	0.24184	0.15189	$u_2 \succ u_3 \succ u_4 \succ u_1$
Proposed Cq-ROFWG ($\lambda = 2$)	0.78139	0.89657	0.84546	0.83306	$u_2 \succ u_3 \succ u_4 \succ u_1$

and complex neutrosophic environments. Such extensions would provide greater flexibility for representing higher levels of uncertainty, hesitation, and inconsistency in practical decision-making problems.

2. *Development of New Aggregation Operators:* The proposed weighted arithmetic and geometric aggregation operators can be further generalized by incorporating advanced aggregation mechanisms such as Bonferroni mean, Heronian mean, Hamy mean, Maclaurin symmetric mean, Dombi, Aczel–Alsina, Einstein, Frank, Schweizer–Sklar, and power aggregation operators. These operators may capture interrelationships among attributes more effectively.
3. *Integration with Advanced MAGDM Approaches:* Future work may combine the proposed Cq-ROF aggregation framework with advanced multi-criteria decision-making techniques, including TOPSIS, VIKOR, EDAS, MARCOS, CODAS, ELECTRE, PROMETHEE, TODIM, and regret theory-based approaches to improve ranking robustness under complex uncertainty.
4. *Consensus-Based Large-Scale Group Decision-Making:* The present study considers a conventional group decision-making environment. Future research may investigate consensus-reaching processes, trust-based expert weighting, social network analysis, and large-scale group decision-making models under the Cq-ROF framework.
5. *Entropy, Similarity, and Distance Measures:* New entropy measures, divergence measures, similarity indices, correlation coefficients, and distance metrics can be developed for Cq-ROF information to enhance clustering, classification, pattern recognition, and uncertainty quantification.
6. *Dynamic and Temporal Decision Models:* Real-world socio-economic systems continuously evolve over time. Therefore, dynamic Cq-ROF decision-making models capable of incorporating temporal information and continuously changing expert evaluations represent an important direction for future research.
7. *Artificial Intelligence and Optimization Integration:* Future investigations may integrate the pro-

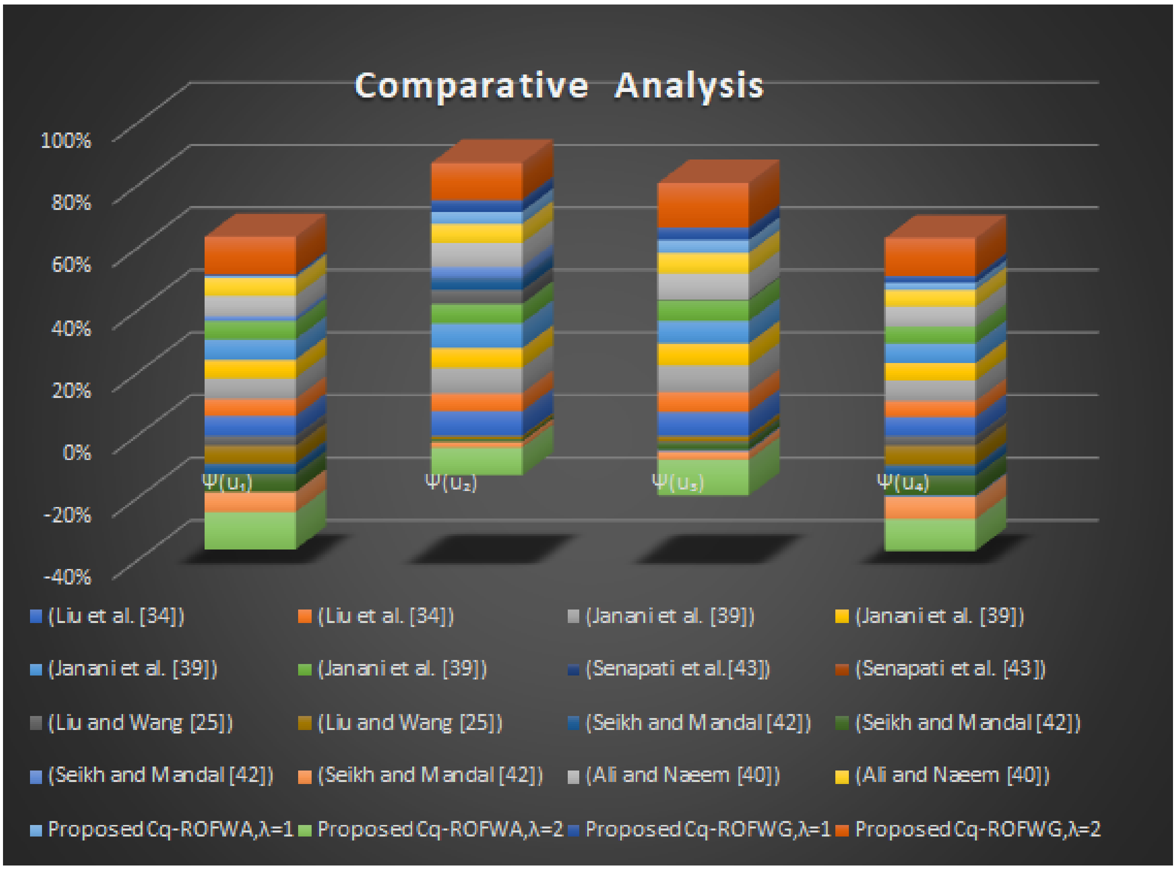

Fig. 5. The graphical representation of Table 16

posed aggregation operators with machine learning, deep learning, evolutionary optimization, reinforcement learning, and intelligent decision-support systems to improve prediction accuracy and adaptive decision-making under complex uncertain environments.

8. *Broader Sustainable Development Applications:* Although the proposed model is validated through socio-economic development in Chinese educational systems, it can be extended to numerous sustainability-related applications, including smart education, healthcare resource allocation, renewable energy planning, supply chain management, financial investment analysis, cybersecurity risk assessment, environmental management, and smart city governance.
9. *Computational Optimization and Software Development:* Future work may focus on improving computational efficiency through parallel algorithms, optimization strategies, and dedicated software packages for implementing Cq-ROF aggregation operators in large-scale practical applications.

6.4 Limitations of the Proposed Work

Although the proposed Complex q-Rung Orthopair Fuzzy (Cq-ROF) aggregation framework provides an effective approach for handling complex-valued uncertainty in multi-attribute group decision-making, the present study has several limitations that should be acknowledged.

1. *Computational Complexity:* The proposed aggregation operators involve multiple complex-valued algebraic operations and parameterized calculations, which may increase computational complexity when dealing with large-scale decision problems involving numerous criteria, alternatives, and decision-makers.
2. *Static Decision-Making Environment:* The proposed MAGDM framework is developed under a static decision environment, assuming that expert evaluations remain unchanged during the decision process. Consequently, it may not adequately capture dynamic situations where decision

information and expert opinions evolve over time.

3. *Limited Validation Scope:* The effectiveness of the proposed methodology is demonstrated through a socio-economic investment assessment in the Chinese educational system. Although this case study verifies the applicability of the model, additional validation across diverse real-world domains would further strengthen its generalizability and practical reliability.

6.5 Implications of the Proposed Study

The proposed study provides significant theoretical, practical, and managerial implications for the development and application of complex fuzzy decision-making methodologies.

1. *Theoretical implications:* From a theoretical perspective, this work extends the framework of Complex q-Rung Orthopair Fuzzy Sets (Cq-ROFSs) by introducing a new family of algebraic operational laws and corresponding weighted aggregation operators. The proposed Cq-ROFWAM and Cq-ROFWGM operators preserve both the amplitude and phase information of complex-valued fuzzy data while satisfying essential mathematical properties, including idempotency, boundedness, monotonicity, and commutativity. Furthermore, the proposed MAGDM algorithm broadens the applicability of Cq-ROFS theory by providing a systematic decision-making framework capable of handling complex-valued uncertainty in multi-expert and multi-attribute environments. Consequently, this study enriches the existing literature on complex fuzzy aggregation and establishes a foundation for future developments of parameterized aggregation operators under more generalized fuzzy environments.
2. *Practical implications:* From a practical standpoint, the proposed decision-making algorithm offers an effective computational tool for solving multiple attribute group decision-making (MAGDM) problems involving uncertain, imprecise, and complex-valued information. The parameterized aggregation mechanism enables decision-makers to incorporate different preference attitudes while preserving the original characteristics of the evaluation information. Since many real-world decision problems involve uncertainty and conflicting assessments from multiple experts, the proposed framework can be readily applied to practical domains such as investment evaluation, supplier selection, medical diagnosis, risk assessment, sustainable development, engineering management, environmental assessment, and intelligent decision-support systems.
3. *Managerial implications:* From a managerial perspective, the proposed framework assists decision-makers by providing a transparent and reliable procedure for ranking competing alternatives under uncertain environments. In the investment evaluation case study considered in this paper, the proposed approach enables investors and policy makers to identify the most suitable investment alternative by simultaneously considering multiple evaluation criteria and expert opinions. The proposed ranking mechanism reduces subjective bias, improves the consistency of decision outcomes, and supports more rational resource allocation. Therefore, the proposed methodology serves as a valuable decision-support tool for managers, investors, and organizational leaders who must make strategic decisions in the presence of complex and uncertain information.

Overall, the proposed Cq-ROFS-based aggregation framework not only advances the theoretical development of complex fuzzy set theory but also provides a flexible and practical decision-support methodology with broad applicability across diverse real-world multiple attribute group decision-making problems.

7. Conclusion

This study introduced a novel family of aggregation operators based on A-complex operational laws within the framework of Complex q-Rung Orthopair Fuzzy Numbers (Cq-ROFNs). Specifically, the Complex q-ROF Weighted Arithmetic Mean Aggregation Operator (Cq-ROFWAMAO), Complex q-ROF Ordered Weighted Arithmetic Mean Aggregation Operator (Cq-ROFOWAMAO), Complex q-ROF

Weighted Geometric Mean Aggregation Operator (Cq-ROFWGMAO), and Complex q-ROF Ordered Weighted Geometric Mean Aggregation Operator (Cq-ROFOWGMAO) were developed to effectively aggregate complex-valued uncertain information in multi-attribute group decision-making (MAGDM) problems. The proposed operators satisfy important mathematical properties, including idempotency, boundedness, and monotonicity, ensuring reliable and consistent aggregation behavior.

To demonstrate the applicability of the proposed methodology, a real-world investment assessment problem in the Chinese educational system was investigated. The obtained results indicate that the proposed framework effectively captures both the amplitude and phase information of uncertain evaluations, producing stable alternative rankings and reliable decision outcomes. Comparative analysis further demonstrates that the proposed approach performs competitively with existing aggregation models while providing greater flexibility for representing complex uncertain information.

From a practical perspective, the proposed decision-making framework can assist policymakers, educational planners, and government agencies in evaluating investment strategies, allocating educational resources, prioritizing development projects, and supporting sustainable socio-economic development under uncertain environments. By incorporating complex-valued expert assessments, the proposed methodology provides a more comprehensive decision-support mechanism for addressing challenging educational planning problems.

Despite these advantages, several limitations should be acknowledged. First, the proposed aggregation operators involve relatively complex mathematical computations, which may increase computational effort for large-scale decision problems. Second, the quality of the final decision depends on the reliability and consistency of expert evaluations, making subjective judgments an inherent limitation of the proposed MAGDM framework. Third, the proposed methodology has been validated using a single educational investment case study, and therefore broader empirical validation across different educational systems and application domains would further strengthen its generalizability.

Future research may focus on validating the proposed framework using real-world datasets from different national educational systems and comparing its performance with classical MCDM approaches such as TOPSIS, VIKOR, PROMETHEE, and ELECTRE. In addition, the proposed Cq-ROF aggregation operators may be extended to dynamic and large-scale group decision-making environments, integrated with intelligent optimization and machine learning techniques, and applied to broader sustainability-related problems including smart education, healthcare management, renewable energy planning, supply chain management, and public policy evaluation.

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Conflicts of Interest

The authors declare no conflicts of interest.

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